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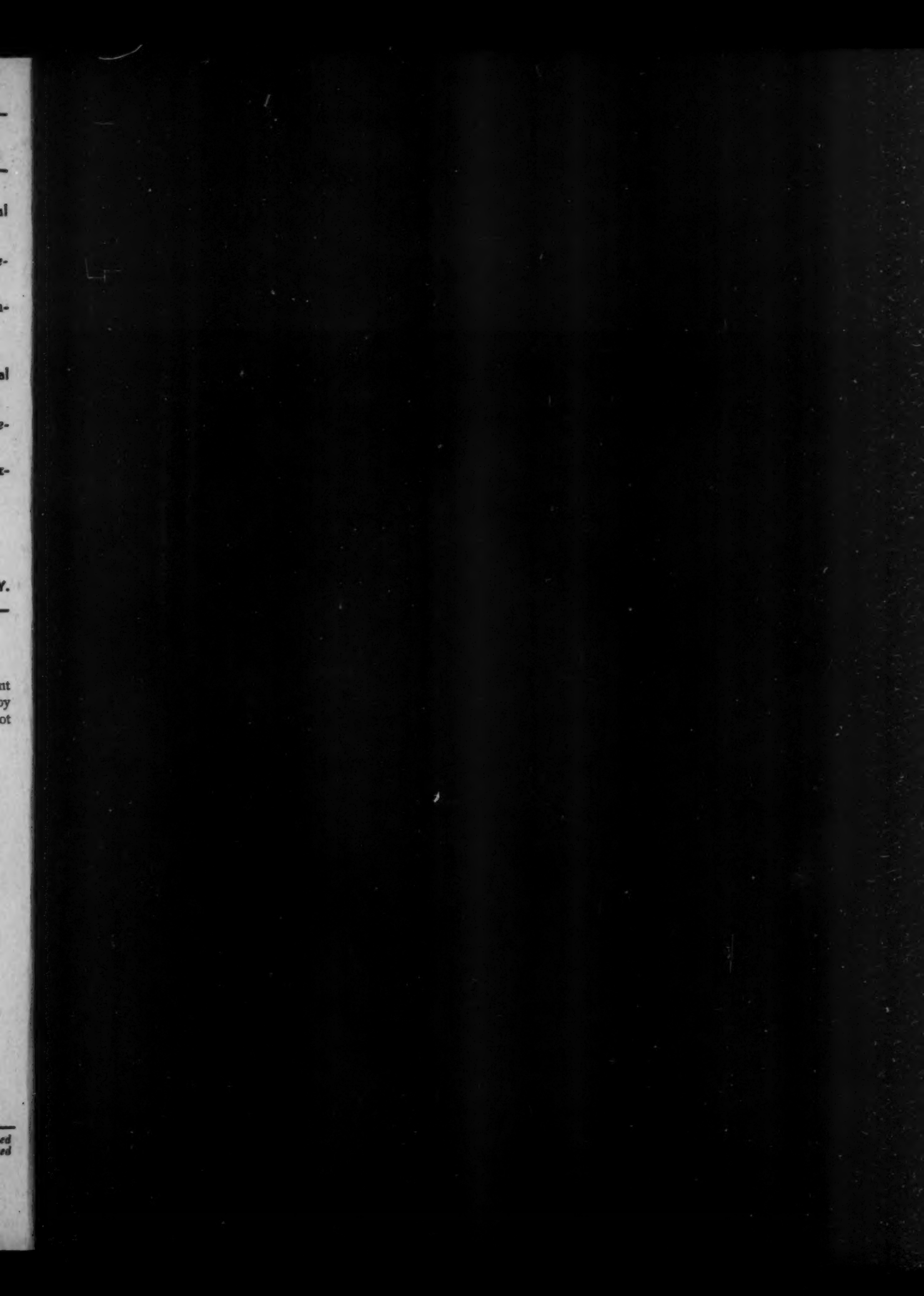

Mathematical Reviews

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FOUNDATIONS

Götlind, Erik. An axiom system for the calculus of propositions. Norsk Mat. Tidsskr. 29, 1–4 (1947). (Swedish)

The author shows that the following axioms are sufficient for the calculus of propositions: (a) $p \vee p \rightarrow p$, (b) $p \rightarrow p \vee q$, (c) $p \vee q \rightarrow q \vee p$, (d) $p \rightarrow r . \rightarrow : q \vee p . \rightarrow . r \vee q$. This is a modification of the axiom system given by D. Hilbert and W. Ackermann [Grundzüge der theoretischen Logik, Springer, Berlin, 1928]. He raises the question whether the new axioms are independent but does not give a complete solution of this problem. *B. Jónsson* (Providence, R. I.).

Sesmat, Augustin, et Lalan, Victor. Équations dans le corps de Boole et relations entre propositions. C. R. Acad. Sci. Paris 224, 1043–1045 (1947).

A discussion of the relation between sentential functions and polynomials over a four-element Boolean algebra. *B. Jónsson* (Providence, R. I.).

Novikov, P. S. On logical paradoxes. Doklady Akad. Nauk SSSR (N.S.) 56, 451–453 (1947). (Russian)

As D. A. Bochvar has shown [Rec. Math. [Mat. Sbornik] N.S. 15 (57), 369–384 (1944); these Rev. 7, 46], the calculus of predicates without distinction of types (system A) is non-contradictory if no definitions of individual predicates are admitted. A definition of a predicate may be replaced by an additional axiom of the form (B):

$$(\mathrm{E}p)(x_1) \cdots (x_n)[p(x_1, \cdots, x_n) \sim G(x_1, \cdots, x_n)],$$

where G contains no free variables other than $x_1, \cdots, x_n$. An occurrence of a variable in a formula is called interior if it stands in the field of an elementary predicate; in the other case the occurrence is exterior. A function G is said to satisfy condition C if each variable has in G either only interior or only exterior occurrences. Now the system A remains non-contradictory after the adjunction of a finite number of axioms (B), provided every function G in them satisfies condition C. A system originating from A by such an adjunction is a system (T). Another form of an axiom defining a predicate p is (D): $p(x_1, \cdots, x_n) \sim G(x_1, \cdots, x_n)$; this is of the form (S) if G satisfies condition C. If $n = 1$, G may be written in the form $G_1 . x(x) . \vee . G_2 . \bar{x}(x) . \vee . G_3$; by simple conventions this form becomes univocal. Then $G_1 \vee G_2 \vee G_3$ is called the "paradoxical consequence" of (D). A system (T) remains noncontradictory after the adjunction of a finite number of axioms of the form (S), provided each of their paradoxical consequences is derivable in a suitable system (T). For a predicate of one variable a necessary and sufficient condition that it leads to contradiction is derived. No elaborate proofs are given. *A. Heyting* (Amsterdam).

Moisil, Gr. C. Contributions à l'étude des logiques non chrysippiennes. IV. Sur la logique de M. Becker. C. R. Acad. Sci. Roum. 7, 9–11 (1945).

Le calcul des modalités de Becker [Jahrbuch Philosophie Phänomenol. Forsch. 11, 497–548 (1930), en particulier, pp. 521–526] admet six modalités positives et six modalités négatives. Il n'est réalisable dans aucune des lattices L_1, L_2 ou L_3 [Moisil, Disquisit. Math. Phys. 1, 307–314 (1941); ces Rev. 8, 557]. *A. Heyting* (Amsterdam).

***Moisil, Grigore Constantin. L'algebra e la logica.** Atti Convegno Mat. Roma 1942, pp. 143–152 (1945).

Ceretheli, Savle. On the character of proof in Aristotle's syllogistic. Trav. Univ. Tbilissi 24, 1–16 (1945). (Georgian. Russian summary)

Borel, Émile. Les paradoxes de l'axiome du choix. C. R. Acad. Sci. Paris 224, 1537–1538 (1947).

It is known that there exists, for a given interval I, a proper subset S of I which is "congruent by decomposition" to the whole of I; more specifically, as may be shown with the aid of the axiom of choice (multiplicative axiom), a proper subset S of I exists such that S is decomposable into $\aleph_0$ disjoint sets which, upon appropriate (independent) translation, pass respectively into disjoint sets whose sum is I. The author adduces this result as ground for rejecting the axiom of choice, in preference, as he contends, to revising the Euclidean axiom of noncongruence (by decomposition) of part and whole in a "finite domain." But this contention does not carry conviction. For it is stretching the Euclidean axiom in question beyond its proper scope to interpret it as admitting decompositions with an infinite number of components. Neither intuition nor experience argue such an extension as of axiomatic nature. On the other hand, the axiom of choice is potently of axiomatic character. Furthermore, congruence (by decomposition) of whole and part in a finite domain, in case an infinite number of components is allowable, may be readily exemplified without the use of the axiom of choice, thus nullifying the author's argumentation. For instance, let T consist of $\aleph_0$ segments which are congruent and parallel to a given horizontal interval, and at distances $1/n$, $n = 1, 2, \cdots, \infty$, directly above it. Then T is congruent (by decomposition) to the set consisting of these segments with any one of them omitted. It may be added, too, that a square is obviously congruent (by decomposition) to half of it if decompositions into c components are admissible. *H. Blumberg.*

Borel, Émile. Sur les difficultés des définitions asymptotiques. C. R. Acad. Sci. Paris 224, 1597–1599 (1947).

An "asymptotic" property of an infinite sequence $\{x_n\}$ (of real numbers, say) is (to speak intuitively) one having reference to its "late elements;" as, for example, the property that x_n is positive for an infinite number of the n. The author makes a number of semi-philosophical observations concerning the difficulties, as he conceives them, in formulating acceptable definitions of sets in terms of asymptotic properties. He intermingles questions of logic with practical considerations, challenges to solve unsolved problems, and personal appraisals of relative significance. In particular,

he ties his observations in closely with the main conclusion of his earlier note [see the preceding review]. These observations may consequently be interpreted, in good part, in relation to this conclusion, which, as the reviewer has shown, is untenable. *H. Blumberg* (Columbus, Ohio).

Andrade Guimarães, A. From the integers to the rational numbers. Cadernos de Análise Geral, no. 20 = Bol. Soc. Portuguesa Mat. Sér. A. 1, no. 1 (1947). x+24 pp. (Portuguese)

An expository lecture, with a preface by Ruy Luís Gomes and historical notes by Luiz Neves Real.

Fraenkel, Abraham A. The recent controversies about the foundation of mathematics. Scripta Math. 13, 17–36 (1947).

Weil, André. L'avenir des mathématiques. Bol. Soc. Mat. São Paulo 1, 55–68 (1946).

***Severi, Francesco. Matematica e matematici d'oggi.** Atti Convegno Mat. Roma 1942, pp. 15–23 (1945).

Dugas, René. L'incompréhension mathématique. Thalès 4, 168–183 (1940).

Cazin, Michel. Algorithme et construction d'une théorie unifiante. C. R. Acad. Sci. Paris 224, 805–807 (1947).

This deals with the role of algorithms in a general procedure proposed by the author for unifying a given set of physical theories. *O. Frink* (State College, Pa.).

Hadamard, J. Les diverses formes et les diverses étapes de l'esprit scientifique. Thalès 4, 23–27 (1940).

ALGEBRA

***Armsen, Paul, und Rohrbach, Hans. Sequenzen in Permutationen.** Ber. Math.-Tagung Tübingen 1946, pp. 36–37 (1947).

The number of permutations of elements 1 to n having exactly x of the $n-1$ successions 12, 23, $\cdots$, $(n-1)n$ is determined, reaching results also given by H. Hadwiger [Mitt. Verein. Schweiz. Versich.-Math. 46, 105–109 (1946); these Rev. 7, 406]. *J. Riordan* (New York, N. Y.).

Götlind, Erik. A combinatorial problem. Mat. Tidsskr. A. 1946, 23–25 (1946). (Swedish)

Consider all purely periodic arrangements with repetition of d out of the g elements $a_1, \cdots, a_g$. (For example, *abab* and *bbbb* are periodic, but *abbb* is not.) Letting such arrangements correspond to g-adic expansions of period d, the author establishes a one-to-one correspondence between the arrangements and the fractions b/a, where $b<a$, $(a, b)=1$, and g belongs to the exponent d modulo a. *W. Feller.*

Thomas, Joseph Miller. Eliminants. Amer. J. Math. 69, 592–598 (1947).

Given a pair of polynomials $x=a_0t^m+a_1t^{m-1}+\cdots+a_m$, $y=b_0t^n+\cdots+b_n$, the eliminant $E(x, y)$ of the polynomials $a_0t^m+\cdots+a_m-x$ and $b_0t^n+\cdots+b_n-y$ is either an irreducible polynomial or a power of an irreducible polynomial, $E(x, y)=A^l$. If a polynomial u of degree p in t can be found such that x and y can be expressed parametrically in terms of u, then the parameter t is said to be reducible and $E(x, y; t)=C[E(x, y; u)]^p$. If $E(x, y; u)$ is the lth power of an irreducible polynomial then $k=pl$. The fundamental theorem demonstrates that the eliminant is reducible if and only if the parameter is reducible. Eliminating by the division process may introduce an extraneous factor but elimination by the above defined eliminant never does. A condition is obtained that the parameter for a single polynomial should be reducible. *D. E. Littlewood.*

Meserve, Bruce Elwyn. Inequalities of higher degree in one unknown. Amer. J. Math. 69, 357–370 (1947).

Given a system S of s polynomials $g_i(x)$ and a finite or infinite interval $a<x\leqq b$, the problem is to examine the segments of the interval in which all $g_i(x)>0$. These segments form the content of system S. Let a polynomial curve $g(x)$ have simple zeros at $\alpha_1, \cdots, \alpha_n$; $g(x)$ will have alternating signs in successive intervals (α_{i-1}, α_i). For each interval (α_{i-1}, α_i) in which $g_i(x)>0$, consider a parabola $-(x-\alpha_{i-1})(x-\alpha_i)=-x^2+2bx-c$, $b^2-4c>0$. Then segment (α_{i-1}, α_i) is the content of the parabola and the content of $g(x)$ is the sum of contents of the set of parabolas, plus an adjustment at one or both extremes, if necessary. If the polynomials $g_i(x)$ and their real zeros are known, with proper multiplicities, it is easy to give a graphical or arithmetical method for finding the segments forming the content of the system S, and to determine their number.

The paper determines these quantities using only the kind of knowledge to be derived from theorems of the Sturm type. Accordingly, the bulk of results obtained involve only rational operations. For example, the number k of segments on which $f(x)>0$ is $\frac{1}{2}[N_0+e(-\infty)+e(+\infty)]+\mu$, where N_0 is the number of real zeros of $f(x)$ of odd multiplicity, $e(x)=0$ for $f(x)\leqq 0$, $e(x)=1$ for $f(x)>0$, μ is the number of "zero minima" of $f(x)$; the number of segments of a finite system $g_i(x)$, $i=1, \cdots, s$, is $k=\sum_{i=1}^{s}k_i-s+1$, where k_i is the number of segments of $g_i(x)$; this bound can be attained.

"Mixed systems," in which one of the $g_i(x)=0$, are considered. To obtain exact results, careful consideration of multiplicities of zeros and of closed and open intervals is required. *A. J. Kempner* (Boulder, Colo.).

Finan, E. J., and McRae, V. V. Equations invariant under root powering. Math. Mag. 21, 29–34 (1947).

Mignosi, Giuseppe. Risoluzione apiristica dei sistemi di equazioni algebriche nei corpi finiti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 250–257 (1947).

An "apiristica" solution of the equations $f(x, y)=p$, $g(x, y)=q$ in a finite field, where p and q are parameters, is of the form $x=A(p, q)$, $y=B(p, q)$, where A and B are polynomials. Such a solution is easily obtained by an interpolation formula, but a prior knowledge of the roots for each p and q seems to be required. *C. C. MacDuffee.*

Cavallucci, Leopoldo. Segnatura di una matrice in un campo di razionalità. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 1, 263–273 (1940).

This is an extension to a general field of the Weyr theory of similarity of matrices with complex elements. This extension has been made several times before, first by Krull [Über Begleitmatrizen und Elementarteilertheorie, Freiburg dissertation, 1921]. *C. C. MacDuffee* (Madison, Wis.).

Van Bergen, F. Sur la représentation des nombres hypercomplexes par des matrices. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 122–137 (1947).

To the extent that it is correct, the material in the first six paragraphs is known [MacDuffee, Monatsh. Math. Phys. **48**, 293–313 (1939); these Rev. **1**, 100]. Then, in seeking real division algebras, the author assumes that the right (left) determinant of the general number is a power of a positive definite quadratic form, and for $n=2, 4$, respectively, he obtains the complex field and the quaternions. This is weaker than Frobenius' theorem [J. Reine Angew. Math. **84**, 1–63 (1877), p. 59] that these are the only real associative division algebras. He obtains the enveloping algebra of quaternions by a method already found by J. A. Ward [Duke J. Math. **7**, 233–248 (1940); these Rev. **2**, 122] and remarks that these 16 matrices are essentially the Dirac matrices. *C. C. MacDuffee* (Madison, Wis.).

Abstract Algebra

Chittenden, E. W. On the number of paths in a finite partially ordered set. Amer. Math. Monthly **54**, 404–405 (1947).

Let P be a finite partially ordered set with initial elements $a_1, a_2, \cdots, a_m$ and terminal elements $b_1, b_2, \cdots, b_n$. Let x_{ij} be the number of paths (maximal simply ordered subsets) of P from a_i to b_j and let N be the matrix (x_{ij}) $(i=1, 2, \cdots, m;$ $j=1, 2, \cdots, n)$. Define the rank of each a_i to be zero and the rank of any other element to be one more than the highest rank of any of its predecessors. Then new elements can be added to P without affecting the x_{ij} so that each path becomes a chain of elements increasing in rank by steps of one from zero to some k, constant for all paths. By simple induction it then follows that N is the product of the incidence matrices which give the number of paths from elements of a given rank to elements of the next higher rank. *I. Halperin* (Kingston, Ont.).

Grau, A. A. Ternary Boolean algebra. Bull. Amer. Math. Soc. **53**, 567–572 (1947).

By a ternary operation in a set K is meant any function of three variables in K whose values are also in K. Postulates for a ternary operation a^bc are introduced which correspond to the properties of the Boolean operation $(a \cap b) \cup (b \cap c) \cup (c \cap a)$, and it is shown that, if for any fixed $p \in K$ the symbols $\cap$ and $\cup$ are defined by $a \cup b = a^pb$ and $a \cap b = a^{p'}b$, then the system so obtained is a Boolean algebra with p as universe element and p' as null element. There is in consequence a one to one correspondence between distinct ternary algebras and abstract Boolean algebras. It is finally shown that $(a \cap b) \cap (b \cup c) \cap (c \cup a)$ is the only realization of the ternary relation a^bc in a Boolean algebra. *H. Wallman* (Cambridge, Mass.).

Bernstein, B. A. Weak definitions of field. Duke Math. J. **14**, 475–482 (1947).

The author gives two sets of postulates for a field, each of which is weaker than the usual sets used to define a field. In one of these sets the ordinary binary operation $+$ is reduced to a unary operation. The postulates of each set are proved independent. *N. H. McCoy.*

van der Waerden, B. L. Divisorenklassen in algebraischen Funktionenkörpern. Comment. Math. Helv. **20**, 68–109 (1947).

Suppose that K is an algebraic function field of one variable with a perfect coefficient field. Let g denote the genus of K. The author discusses the field of Abelian functions belonging to K. An important tool is the construction of a projective model M of the Abelian function field. Let $\mathfrak{A}=\mathfrak{p}^{(1)} \cdots \mathfrak{p}^{(g)}$ be a divisor of degree g on K. Using a projective model, free from singularities, Γ of K in a space of sufficiently high dimension, the representation of $\mathfrak{A}$ as a group of points is used to determine the u-form of $\mathfrak{A}$ in the customary fashion. The coefficients $a_0, \cdots, a_h$ of the u-form are termed the Chow coordinates of $\mathfrak{A}$. Thus, all divisors $\mathfrak{A}$ determine in the h-dimensional projective space an irreducible algebraic manifold M. This manifold is asserted to be free from singularities. (In the proofs the author's theory of faithful specialization is much used; multiplicity of a point on M is defined as the intersection multiplicity with a general linear subspace of complimentary dimension.) Next suppose that A is a divisor class of degree 0, and let $\mathfrak{P}/\mathfrak{D}$, with integral divisors of degree g, be a divisor of A. Using elimination theory the equivalence $\mathfrak{P}/\mathfrak{D} \sim \mathfrak{P}'/\mathfrak{D}'$ is expressed by a homogeneous system of equations involving the Chow coordinates of the four divisors. In order to construct an algebraic variety $\mathfrak{K}$ whose points are in one-to-one correspondences with the classes A of K, the totality of all representations $(\mathfrak{R}, \mathfrak{S})$ of the class $A=(\mathfrak{P}/\mathfrak{D})=(\mathfrak{S}/\mathfrak{R})$ is considered. The algebraic description of divisor equivalence then leads to the fact that the sets $(\mathfrak{R}, \mathfrak{S})$ make up an algebraic manifold N_A on the manifold N of all pairs for variable A. Finally it is shown that all N_A form an algebraic system $\mathfrak{K}$. The manifold describing $\mathfrak{K}$ has the property that its points are in one-to-one correspondence with the classes A. Moreover, the divisor relations $C=AB$ and $A=\mu B$, where μ is a regular multiplication of the Abelian function field, are described by equations involving the Chow coordinates of the corresponding points. The decomposition theory of points on $\mathfrak{K}$ (faithful specializations) in covering varieties without singularities is applied to show that each equation $A=\mu B$, A is given, has $n(\mu)$ solutions, where $n(\mu)$ is the reduced degree of the field of $\mathfrak{Y}$ over the field of $\mathfrak{X}$; here $\mathfrak{X}, \mathfrak{Y}$ are generic divisors of degree g for which $\mathfrak{X}/\mathfrak{D}=\mu(\mathfrak{Y})/\mathfrak{A}$ with fixed $\mathfrak{D}$ and $\mathfrak{A}$. [It seems to the reviewer that the author requires for the preceding results a proof that the variety K, and some of its algebraic extensions, are free from singularities.] *O. F. G. Schilling* (Chicago, Ill.).

Snapper, Ernst. Polynomial matrices in one variable, differential equations and module theory. Amer. J. Math. **69**, 299–326 (1947).

In dieser Arbeit, die sich stark auf die Arbeit von B. M. Grundy [Proc. Cambridge Philos. Soc. **38**, 241–279 (1941); diese Rev. **4**, 129] stützt, wird die Theorie der Elementarteiler von Polynommatrizen in die Sprache der Modultheorie übersetzt und daran anschliessend eine Verallgemeinerung gegeben. Ordnen wir einer $m \times s$-Matrix A mit Koeffizienten aus dem Polynombereich $\mathrm{P}[x]$ über dem Grundkörper P den aus den Spaltenvektoren der Matrix erzeugten $\mathrm{P}[x]$-Modul M zu, so ist der Restklassenmodul des Moduls V aller m-reihigen Vektoren mit Komponenten aus $\mathrm{P}[x]$ modulo M invariant über $\mathrm{P}[x]$ bezüglich der Multiplikation von A mit unimodularen Polynommatrizen (Klasseneinteilung!) und umgekehrt gehört zu jedem aus V entstehendem Faktormodul genau eine Klasse von Matrizen.

Im zweiten Teil der Arbeit werden allgemein Moduln V, die einen Ring $\mathfrak{o}$ mit Einheitselement 1 als Operatoren besitzen, so dass $1 \cdot V = V$ ist und für die in V enthaltenen $\mathfrak{o}$-Moduln $M, M_1, M_2, \cdots$ der Maximalkettensatz gilt, betrachtet. Wenn $M_1 \subseteq M_2 \subseteq V$, so wird mit $M_1 : M_2$ das Ideal, bestehend aus allen Elementen ξ aus $\mathfrak{o}$, für die $\xi M_2 \subseteq M_1$ ist, definiert. Das Ideal $\mathfrak{f}(M) = M : V$ heisst das Fundamentalideal von M. Als Radikal $\mathfrak{r}(M)$ wird das Radikal $\mathfrak{r}(\mathfrak{f})$ von $\mathfrak{f}$, das aus allen mod $\mathfrak{f}$ nilpotenten Elementen von $\mathfrak{o}$ besteht, bezeichnet. Der Modul M heisst primärer Teilmodul von V, wenn aus $\alpha v \in M$, $\alpha \notin \mathfrak{r}(M)$ die Relation $v \in M$ folgt. Satz 2.11: Wenn M primär ist, so ist das Ideal $\mathfrak{f}(M)$ primär, $\mathfrak{r}(\mathfrak{f})$ ist ein Primideal $\mathfrak{p}$. Der Modul M ist genau dann $\mathfrak{p}$-primär, wenn (1) aus $\alpha v \in M$, $v \notin M$ folgt $\alpha \in \mathfrak{p}$; (2) $\mathfrak{f}(M) \subseteq \mathfrak{p}$; (3) für jedes Element aus $\mathfrak{p}$ eine Potenz in $\mathfrak{f}(M)$ liegt.

Eine unverkürzbare Darstellung von M als Durchschnitt $\mathfrak{p}_j$-primärer Moduln M_j mit lauter verschiedenen Primidealen $\mathfrak{p}_j$ heisst eine Noether-Zerlegung von M. Satz 2.21: Es gibt stets eine Noetherzerlegung, die Primideale $\mathfrak{p}_j$ sind bis auf die Reihenfolge eindeutig bestimmt und heissen die assoziierten Primstellen von M. Das Radikal von M ist gleich dem Durchschnitt der Radikale der Noetherkomponenten M_j.

Als Komponente von M werden auch die Durchschnitte einiger Primärkomponenten einer beliebigen, aber festen Noetherzerlegung von M bezeichnet. Die Komponente heisst isoliert, wenn jede zu M assoziierte Primstelle, die in einer assoziierten Primstelle der Komponente enthalten ist, selbst zu der Komponente assoziiert ist. Die isolierten Komponenten sind eindeutig bestimmt durch ihre assoziierten Primstellen. Satz 2.41: Zu jedem $\mathfrak{p}$-primären Modul Q gibt es eine $\mathfrak{p}$-primäre Kompositionsreihe $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_{l-1} \subset Q_l = V$ von Q nach V, so dass die Q_i $\mathfrak{p}$-primär sind und sich zwischen Q_i und Q_{i+1} kein $\mathfrak{p}$-primärer Modul einschieben lässt. Alle diese Kompositionsreihen haben dieselbe Länge l, die Länge von Q. Jede mit Q beginnende bis V aufsteigende Folge $\mathfrak{p}$-primärer Moduln lässt sich zu einer $\mathfrak{p}$-primären Kompositionsreihe verfeinern. Für einen beliebigen Modul M mit assoziierter Primstelle $\mathfrak{p}$ werde mit $M(\mathfrak{p})$ die isolierte Komponente, deren assoziierte Primstelle in $\mathfrak{p}$ enthalten sind, bezeichnet, ferner mit $M'(\mathfrak{p})$ die isolierte Komponente, deren assoziierte Primstellen eigentlich in $\mathfrak{p}$ enthalten sind. Hilfssatz 2.51: $M(\mathfrak{p})$ ist $\mathfrak{p}$-primär in $M'(\mathfrak{p})$ enthalten, nämlich als die isolierte $\mathfrak{p}$-primäre Komponente von M betrachtet als Teilmodul von $M'(\mathfrak{p})$. Das $\mathfrak{p}$-primäre Ideal $\mathfrak{e}(\mathfrak{p}) = M(\mathfrak{p}) : M'(\mathfrak{p})$ heisst $\mathfrak{p}$-Elementarteiler von M und sein Exponent heisst $\mathfrak{p}$-Exponent von M. Die Länge $l(\mathfrak{p})$ von $M(\mathfrak{p})$ als $\mathfrak{p}$-primärer Teilmodul von $M'(\mathfrak{p})$ heisst die $\mathfrak{p}$-Länge von M. Wenn M aus den Spaltenvektoren einer Polynommatrix erzeugt wird, wie vorhin angegeben, dann ist der grösste von Null verschiedene Elementarteiler gleich dem Produkt aller $\mathfrak{p}$-Elementarteiler mit $\mathfrak{p} \neq 0$ ($\mathfrak{p}$-Exponenten stimmen überein!), während das Produkt aller von Null verschiedenen Elementarteiler gleich dem Produkt der von Null verschiedenen Primideale $\mathfrak{p}$ potenziert mit ihrer Länge ist. Schliesslich ist für jede von Null verschiedene Primstelle $M'(\mathfrak{p})$ gleich der Hülle von M, die aus allen jenen Elementen x aus V besteht, für die ein Vielfaches ξx in M liegt ($\xi \neq 0$).

Längenkriterium: Zwei Moduln $M_1 \subseteq M_2$ sind genau dann gleich, wenn sie dieselben assoziierten Primstellen und dieselben $\mathfrak{p}$-Längen besitzen. Weiterhin wird vorausgesetzt, dass der Teilmodul, der aus dem Nullelement allein besteht, primär ist, sodass sein Radikal $\mathfrak{r}$ ein Primideal ist. Als Hülle von M wird der Modul $H(M)$ aller Elemente v von V, für die ein Element ξ aus $\mathfrak{o}$ existiert, das nicht in $\mathfrak{r}$ liegt, so dass $\xi v \in M$, bezeichnet. Als Rang von M wird die Länge $r(M)$ des 0-Moduls, als $\mathfrak{r}$-primärer Teilmodul von M betrachtet, bezeichnet. Um die Lösbarkeit eines Systems $\sum_{j=1}^{s} \alpha_{ij} x_j = \gamma_i$ ($i = 1, \cdots, m$; $\alpha_{ij}, \gamma_i \in \mathfrak{o}$) für die s Unbekannten $z_1, z_2, \cdots, z_s$ zu untersuchen, ordnen wir innerhalb des m-dimensionalen $\mathfrak{o}$-Vektormoduls V, die aus den Spaltenvektoren der einfachen Matrix $A = (\alpha_{ij})$ bzw. der erweiterten Matrix $B = (\alpha_{ij}, \beta_i)$ erzeugten $\mathfrak{o}$-Moduln M bzw. M_1 zu. Nach Satz 2.72 ist das System genau dann lösbar, wenn M und M_1 denselben Rang haben, ferner in den von Null verschiedenen assoziierten Primstellen übereinstimmen und für jede von ihnen dieselbe $\mathfrak{p}$-Länge haben. Daraus ergeben sich alle bekannten Kriterien über die Auflösung von linearen Gleichungen. Eine Anwendung auf die Auflösung linearer Differentialgleichungen mit Polynomen als Koeffizienten wird gegeben. *H. Zassenhaus* (Hamburg).

Stewart, B. M. Left associated matrices with elements in an algebraic domain. Amer. J. Math. **69**, 562–574 (1947).

Let F be an algebraic field of order k over the rational field Ra, and $[F]$ the ring of all integral elements of F. The set of all n-by-n matrices with elements in $[F]$ will be denoted by $[M]$. The author is concerned with the problem of finding conditions that two matrices A and B of $[M]$ shall be left associates in $[M]$, that is, that there exist a unimodular matrix P of $[M]$ such that $PA = B$. If each element of a matrix A of $[M]$ is replaced by its k-by-k second matric representation over Ra, one obtains an enlarged matrix A^E of order kn-by-kn, but with elements in the domain $[Ra]$ of rational integers. The principal result is the theorem that matrices A and B of $[M]$ are left associates in $[M]$ if and only if the corresponding enlarged matrices A^E and B^E are left associates. The actual construction of a transforming matrix P is fairly difficult but is discussed in detail in the second part of the paper. The methods and results are closely related to the fundamental work of Steinitz [Math. Ann. **71**, 328–354 (1911); **72**, 297–345 (1912)]. *N. H. McCoy* (Northampton, Mass.).

Brewer, B. W. Smallest fields over which every polynomial is solvable by radicals. Duke Math. J. **14**, 283–287 (1947).

The field F is termed by the author an S_R-field if each of its algebraic extensions is solvable over F. The field $GF(p, p^n)$ [which is the join of all the Galois fields $GF(p, p^n)$ of characteristic p and absolute degree p^n] is an S_R-field and is part of every S_R-field of characteristic p. Every S_R-field contains a smallest S_R-field; but the field of all the real algebraic numbers contains a continuum of mutually nonisomorphic smallest S_R-fields. *R. Baer* (Urbana, Ill.).

Jacobson, N. A note on division rings. Amer. J. Math. **69**, 27–36 (1947).

Généralisant ses résultats antérieurs [Ann. of Math. (2) **41**, 1–7 (1940); ces Rev. **1**, 198] l'auteur montre qu'il existe une correspondance biunivoque entre les sous-corps L d'un corps non commutatif K tel que K ait une dimension finie comme espace vectoriel à gauche sur L, et les sous-anneaux $\mathfrak{L}$ de l'anneau d'endomorphismes du groupe additif de K, qui contiennent le corps K_r des multiplications à droite par des éléments de K, et sont de dimension finie comme espace vectoriels à droite sur K_r: l'anneau $\mathfrak{L}$ correspondant à L est l'anneau des endomorphismes de K considéré comme espace vectoriel à gauche sur K, et la dimension (à droite) de $\mathfrak{L}$

sur K, est égale à la dimension (à gauche) de K sur L. Ce résultat permet à l'auteur de caractériser les groupes d'automorphismes Γ de K tels que le sous-corps L de K formé des éléments invariants par Γ soit tel que K ait une dimension finie (à gauche) sur L: le sous-groupe Γ' des automorphismes intérieurs contenus dans Γ doit être d'indice fini n dans Γ, et le sous-corps H de K engendré par les α tels que les automorphismes intérieurs $\xi \to \alpha\xi\alpha^{-1}$ appartiennent à Γ', doit être de rang fini d sur le centre de K. La dimension de K sur L est alors nd, et le groupe $\bar{\Gamma}$ de tous les automorphismes de K laissant invariants les éléments de L s'obtient en composant les automorphismes de Γ avec les automorphismes intérieurs produits par les éléments de H; si $\bar{\Gamma}=\Gamma$, Γ est dit complet. Il y a une correspondance biunivoque entre groupes complets d'automorphismes de K et leurs corps d'invariants; tout corps intermédiaire entre K et le corps des invariants L d'un groupe complet Γ est lui-même corps d'invariants d'un sous-groupe de Γ. [Ces résultats sont identiques à une partie de ceux obtenus simultanément par H. Cartan; voir le compte-rendu suivant.]

J. Dieudonné (Nancy).

Cartan, Henri. Extension de la théorie de Galois aux corps non commutatifs. C. R. Acad. Sci. Paris **224**, 87–89 (1947).

Cartan, Henri. Les principaux théorèmes de la théorie de Galois pour les corps non nécessairement commutatifs. C. R. Acad. Sci. Paris **224**, 249–251 (1947).

The principal theorems of Galois theory, in Artin's formulation, are extended to division rings. Let K be a division ring or sfield, and let G be a group of automorphisms of K. Denote by $J(G)$ the subsfield of K which is generated by those elements $k \in K$ for which the inner automorphism $x \to kxk^{-1}$ belongs to G; G is called complete if the inner automorphisms which are effected by the elements of $J(G)$ belong to G. If L is a subsfield of K, the group $G_{K,L}$ of all automorphisms of K which leave the elements of L fixed is complete. The sfield K is said to be normal over L if the subsfield of elements left fixed by $G_{K,L}$ is L itself. The chief results are the following.

(1) Let G be a group of automorphisms of a sfield K and let L be the sfield of elements left fixed by G. Then K is of finite (left) dimension over L if and only if (a) the factor group of G by the subgroup formed by those inner automorphisms which are in G is of finite order n; (b) $J(G)$ is of finite dimension d over the center of K. Moreover, the left- and right-dimension of K over L are then equal to nd.

(2) If K is normal and of finite dimension over L, then K is normal over any intermediary sfield and there is the usual Galois correspondence between the intermediary sfields between K and L and the complete subgroups of $G_{K,L}$.

(3) Let K and L be as in (2) and let H be a sfield between K and L. Then every isomorphism of H into K which leaves the elements of L fixed can be extended to an automorphism of K. If Γ denotes the group formed by the automorphisms in $G_{K,L}$ which map H into itself, then $G_{H,L}$ is isomorphic with $\Gamma/G_{K,H}$; H is normal over L if and only if $G_{K,L}$ is the smallest complete group containing Γ. In particular, if $G_{K,L}=\Gamma$, i.e. if $G_{K,H}$ is normal in $G_{K,L}$, then H is normal over L.

This theory includes the theorems of Skolem-Noether on automorphisms of central division algebras, as well as Jacobson's results on groups of outer automorphisms of sfields. An interesting result of the theory is the following theorem: if K is of finite dimension over its center C, then the only subsfields of K which contain C and which are mapped into themselves by every inner automorphism of K are K and C. *C. Chevalley* (Princeton, N. J.).

Jennings, S. A. On rings whose associated Lie rings are nilpotent. Bull. Amer. Math. Soc. **53**, 593–597 (1947).

With any ring R one may associate a Lie ring $\mathfrak{R}$ by setting $x \circ y = xy - yx$. By putting $\mathfrak{R}_1=\mathfrak{R}$ and $\mathfrak{R}_k=[\mathfrak{R}_{k-1}, \mathfrak{R}]$, where $k>1$ and $[\mathfrak{R}_{k-1}, \mathfrak{R}]$ is the Lie ideal generated by all elements of the form $x \circ y$ such that $x \in \mathfrak{R}_{k-1}$, $y \in \mathfrak{R}$, one obtains a descending chain of Lie ideals. If $\mathfrak{R}_n=0$ for some n, then R is called L-nilpotent. In continuing his earlier study [Duke Math. J. **9**, 341–355 (1942); these Rev. **4**, 69] the author considers the ideals $\bar{R}_k$ in R generated by the elements of $\mathfrak{R}_k$ and proves that if R is L-nilpotent, then $\bar{R}_2$ is a nil-ideal, while $\bar{R}_3$ is nilpotent. If in addition R is finitely generated, then also $\bar{R}_2$ is nilpotent. The author concludes by showing that a finitely generated nil-ring is nilpotent if and only if it is L-nilpotent. *J. Levitzki* (Jerusalem).

***Grell, H. Über die Erhaltung der Kettensätze der Idealtheorie.** Ber. Math.-Tagung Tübingen 1946, p. 67 (1947).

Let R be an integral domain in which (1) every proper ideal is a product of primary ideals; (2) every primary ideal contains a power of its prime; (3) every prime ideal is maximal; (4) no proper ideal is an irredundant intersection of infinitely many ideals. It is stated without proof that the same properties are possessed by any ring integrally dependent on R and contained in an arbitrary finite extension of the quotient field of R. *I. S. Cohen* (Philadelphia, Pa.).

Kurosh, A. Non-associative free algebras and free products of algebras. Rec. Math. [Mat. Sbornik] N.S. **20**(62), 239–262 (1947). (Russian. English summary)

The author introduces the notion of a nonassociative free algebra with an arbitrary set of free generators; previously only the case where the number of the free generators is finite was considered [R. M. Thrall, Amer. J. Math. **64**, 371–388 (1942); these Rev. **3**, 262]. The author first shows that the cardinal number of the set of the free generators is an invariant which determines the algebra up to isomorphism. His next step is the establishment of the following theorem. Any nonzero subalgebra of a nonassociative free algebra A with one generator is again a nonassociative free algebra. Moreover, the algebra A contains subalgebras with any prescribed finite number of free generators, and also subalgebras whose free generators form an infinite sequence. An important tool for his further investigations is the notion of a nonassociative free product (for short: free product) which is defined as follows. Let A_α be an arbitrary set of not necessarily associative algebras over a field P. In each algebra a basis over P is chosen and the union of all bases is denoted by L. The basis of the free product of the A_α is then formed by all possible nonvoid words (with all possible distributions of brackets) in the elements of L, such that no bracket of length 2 contains two elements belonging to the same algebra. The importance of this concept may be illustrated by the following properties. The free product of the A_α is independent of the choice of the bases in the A_α. Each nonassociative free algebra is the free product of the subalgebras generated by each of its free generators. Each nonassociative free algebra with one generator is nondecomposable. The author's principal result concerning free products is the following theorem. Every nonzero subalgebra of the free product of the algebras A_α is a free product of its

intersections with the factors A_α and possibly of some additional free algebra. As a consequence of this theorem the author obtains the following result. Any nonzero subalgebra of a nonassociative free algebra is also a nonassociative free algebra. Other results are centered around the concept of isomorphic decompositions, which is defined as follows. Two free decompositions of an algebra are said to be isomorphic if they differ only by factors which are free algebras with one generator (of which it can be proved that their number is the same in both decompositions). The author shows that any two free decompositions of an algebra possess isomorphic extensions and hence that all decompositions with nondecomposable factors (whenever such decompositions exist) are isomorphic. In conclusion, the author gives an example of a decomposable algebra which cannot be decomposed into a free product of nondecomposable algebras.

The author notes that it is impossible to develop a similar theory for free associative rings or algebras, or even for free nonassociative rings, and emphasizes the parallelism of his theory to the theory of free products of groups and also to the theory of free products of projective planes due to L. Kopeikina [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 495–526 (1945); these Rev. **8**, 167]. The author remarks that it would be of interest to discover the reason for this parallelism. *J. Levitzki* (Jerusalem).

Arens, Richard. Linear topological division algebras. Bull. Amer. Math. Soc. **53**, 623–630 (1947).

Various characterisations of the real or complex numbers or quaternions among topological division algebras are obtained. Among other results known theorems by Mazur [C. R. Acad. Sci. Paris **207**, 1025–1027 (1938)], Šilov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 83–84 (1940); these Rev. **2**, 314] and Gelfand [Rec. Math. [Mat. Sbornik] N.S. **9** (51), 3–24 (1941); these Rev. **3**, 51] are generalised by omitting the assumption of commutativity or of the existence of a norm or, in the norm case, that the scalars are complex numbers. The problem as to whether the assumption of the continuity of the inverse operation is an essential one is studied carefully in each case. An example of a not-normed ring is given in which the set of nonsingular elements does not form a topological group.

The class A of functions holomorphic on an open subset of the complex plane is studied. This, with the usual topology, is a complete convex and metrizable algebra which is, however, not normable. Although it has no topological zero divisor, i.e., there is no element z in it such that 0 is a limit point of zA or Az for some closed set z not containing 0, it is not a division algebra. The set of singular elements in this algebra is open whereas in a Banach algebra it is always closed. *O. Todd-Taussky* (Princeton, N. J.).

THEORY OF GROUPS

*Speiser, Andreas. **Problemi attuali della teoria dei gruppi astratti.** Atti Convegno Mat. Roma 1942, pp. 85–90 (1945).

Artin, E. Braids and permutations. Ann. of Math. (2) **48**, 643–649 (1947).

The group B_n of braids with n strings [Artin, same Ann. (2) **48**, 101–126 (1947); these Rev. **8**, 367] admits a natural homomorphism onto the group S_n of permutations of n elements. The author proves that if $n \neq 4, 6$ this homomorphism is (modulo inner automorphisms of S_n) the only homomorphism of B_n onto a transitive subgroup of S_n. If $n=4$ or $n=6$ other homomorphisms exist, but still the natural one can be identified by abstract algebraic properties. *S. Eilenberg* (New York, N. Y.).

Paige, L. J. A note on finite Abelian groups. Bull. Amer. Math. Soc. **53**, 590–593 (1947).

La théorie des quasigroupes conduit à la question suivante. Étant donné un groupe fini, est il possible de trouver une application $x \to \theta(x)$ biunivoque telle que $\eta(x) = x\theta(x)$ soit aussi biunivoque? L'auteur montre par un procédé constructif que cela est possible pour un groupe abélien sauf dans le cas où le groupe possède un seul élément d'ordre 2. *J. Kuntzmann* (Grenoble).

Lombardo-Radice, Lucio. Il numero delle componenti irriducibili delle algebre legate a gruppi di ordine finito nel corpo razionale. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 291–304 (1946).

Let G be an Abelian group of order n, and A' its group ring over the rational numbers. The author expresses the number of irreducible components of A' as a sum involving the ϕ-function of divisors of n. To prove this, he considers the primitive idempotents of the group ring A over the complex numbers, which he exhibited in a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 686–690 (1946); these Rev. **8**, 252]. Grouping these by symmetry he obtains the primitive idempotents of A'. *I. Kaplansky* (Chicago, Ill.).

Todd, J. A. On the simple group of order 25920. Proc. Roy. Soc. London. Ser. A. **189**, 326–358 (1947).

In his Cambridge Tract [A Locus with 25,920 Linear Self-Transformations, Cambridge University Press, 1946; these Rev. **8**, 400] H. F. Baker discusses at length the geometrical properties of the quartic locus $\sum x_r x_s x_t x_u = 0$ $= x_1 + \cdots + x_6$ in four dimensions, particularly with regard to its 45 nodes. Certain linear spaces are defined which constitute an important configuration in S_4, and this configuration is left invariant under a harmonic inversion with respect to any node A and its polar prime; Baker denotes this operation by $p(A)$. The fundamental relations connecting these transformations were given by Baker who showed that they constitute a single conjugate set of the group G of transformations of the configuration into itself, and so can be taken as a set of generators of G, since G is simple. Todd investigates the remaining operations of the group, defined geometrically and expressed in terms of the $p(A)$. Since the group G is isomorphic to the group G^* of the 27 lines on the general cubic surface in S_3, he is able to relate his work to that of Coxeter, Frame and others, expressing their generators of the group in terms of the $p(A)$. *G. de B. Robinson* (Toronto, Ont.).

Tchounikhin, S. A. Sur les p-propriétés des groupes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 477–480 (1947).

If G is a finite group and p any prime, the author defines the following "p-properties": (i) G is p-Abelian if every element of G of order p^λ, $\lambda \geqq 0$, permutes with every element of G of order prime to p; (ii) G is p-nilpotent if the p-Sylow subgroup of G is normal; (iii) G is p-solvable (p-supersolvable) if the factors in a composition series (chief series) of G have orders either equal to p or prime to p. In par-

ticular, groups whose orders are prime to the given p enjoy all of the above p-properties. The author states that many of the familiar criteria for a group to be solvable, nilpotent, etc., hold with but slight modification for the corresponding p-property. Typical of the theorems announced are the following: G is p-solvable if and only if it has a chain of subgroups $G=G_0\supset G_1\supset\cdots\supset G_l=1$ each normal in the preceding and such that G_{i-1}/G_i is p-Abelian, $i=1, 2, \cdots, l$; G is p-nilpotent if and only if it has a chain of normal subgroups $G=G_0\supset G_1\supset\cdots\supset G_l=1$ such that, for $i=1, 2, \cdots, l$, every element of G_{i-1}/G_i, whose order is prime to p, is permutable with every p-Sylow subgroup of G/G_i. [Misprint: in theorem 11, for "p-développable" read "p-résoluble".]

S. A. Jennings (Vancouver, B. C.).

Frame, J. Sutherland. On the reduction of the conjugating representation of a finite group. Bull. Amer. Math. Soc. **53**, 584–589 (1947).

If the conjugating representation P of a finite group G is defined by assigning to the group element γ the matrix $P(\gamma)=L(\gamma^{-1})R(\gamma)$, where L and R are the left- and right-regular representations of G, the following facts are proved: (a) P is an intransitive permutation representation of G whose transitive constituents P_σ are in one-to-one correspondence with the classes σ of conjugate elements of G; (b) P_σ is the permutation representation generated by the normaliser N_σ of any element of σ; consequently, (c) if the center of G is different from the identity, P cannot be faithful; on the other hand, (d) P is equivalent to the sum of $\bar{\Gamma}_\sigma\times\Gamma_\sigma$, where Γ_σ is the irreducible representation of G associated with the class σ. A unitary transformation is given which completely reduces both L and R and consequently also P. *G. de B. Robinson* (Toronto, Ont.).

Rieger, Ladislav S. On the ordered and cyclically ordered groups. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1946**, no. 6, 31 pp. (1947). (Czech. English summary)

The author discusses simply ordered groups; he was unaware of related recent work [cf. G. Birkhoff, Ann. of Math. (2) **43**, 298–331 (1942); these Rev. **4**, 3]. Thus he rediscovers l-ideals ("magnitude subgroups") and their connection with congruence relations, and lexicographic unions. Among his new results are rather complicated necessary and sufficient conditions on order alone, for an ordered group to be commutative. A special study is made of discrete ordered groups. If G and its factor-groups are all discrete, then G can be decomposed into infinite cycles.

In the English summary, further results are announced. Cyclically ordered (c.o.) groups are defined; every finite c.o. group is a finite cycle. Every c.o. group can be "represented" as a factor-group of the "normal infinite cycle"; c.o. groups can be topologized in a "natural" way; every compact c.o. group is topologically isomorphic with the group of rotations of the unit circle. *G. Birkhoff.*

Eilenberg, Samuel, and MacLane, Saunders. Cohomology theory in abstract groups. II. Group extensions with a non-Abelian kernel. Ann. of Math. (2) **48**, 326–341 (1947).

[For part I see the same vol., 51–78 (1947); these Rev. **8**, 367.] Let ϕ be a homomorphism of a non-Abelian group E onto a group Q, with kernel K. Let $A(K)$ be the group of automorphisms of K, $I(K)$ the inner automorphisms. There exists a natural homomorphism θ: $Q\to A(K)/I(K)$ and Q operates in a natural manner on the center G of K. Then (E, ϕ) is called a group extension of Q by (K, θ). The authors consider a fixed group Q, and a fixed Abelian group G upon which Q operates in a fixed manner. The problem is to determine whether or not a given Q-kernel (K, θ) (i.e., group K with center G, and homomorphism θ: $Q\to A(K)/I(K)$) comes from some group extension of Q; when it does, (K, θ) is called extendible. First, it is shown that the totality of Q-kernels (K, θ) with center G fall into certain similarity classes which form an Abelian group under a certain G-multiplication. Next, each (K, θ) determines a 3-dimensional cocycle F_3 of Q over G [see part I] and (K, θ) is extendible if and only if $F_3=1$. In fact, the group of similarity classes of Q-kernels is isomorphic to the cohomology group $H^3(Q, G)$. When (K, θ) is extendible, the inequivalent extensions of Q by (K, θ) can be put into one-to-one correspondence with the elements of $H^2(Q, G)$. *P. A. Smith.*

Kasner, Edward, and DeCicco, John. Groups of harmonic transformations. Duke Math. J. **14**, 327–338 (1947).

The authors consider the set (H) of harmonic transformations of the real or complex Cartesian plane,

$$X=\varphi(x, y),\quad Y=\psi(x, y),$$
$$\varphi_{xx}+\varphi_{yy}=0,\quad \psi_{xx}+\psi_{yy}=0,$$

for which the Jacobian does not vanish, and seek the groups contained in (H). Their first step is to obtain all the transformations in (H) such that the inverse functions are harmonic. Next the groups in (H) in the imaginary domain are determined, and from these explicit results it is shown that the only groups in (H) in the real domain are the conformal group, the affine group and the subgroups of these two.

E. F. Beckenbach (Los Angeles, Calif.).

Gelfand, I., and Neumark, M. The principal series of irreducible representations of the complex unimodular group. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 3–4 (1947).

For noncompact groups in general, and the complex unimodular group G in particular, not every irreducible representation occurs as a constituent of the regular representation. The authors define the quasi-regular representation of a group as the direct sum of all of the distinct irreducible representations (each with multiplicity one) which appear in the regular representation. The main result of the present note is a characterization of the space for the quasi-regular representation as the Hilbert space of all square summable functions on the subgroup H consisting of all elements of G which considered as matrices have zeros below the main diagonal. *R. M. Thrall.*

Levitan, B. On the theory of unitary representations of locally compact groups. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 407–427 (1946). (Russian. English summary)

Let G be a locally compact topological group satisfying the second denumerability axiom (apparently essential at certain points); let e be its identity element. Let L_p be the class of all complex measurable functions f with integrable $|f|^p$ on G (with respect to the right-invariant Haar measure for G). In the Hilbert space L_2 the kernel $f(xy^{-1})$ defines an integral operator $A=A_f$ which is self-adjoint and of Carleman type when f is Hermitian (that is, $f(x^{-1})=\overline{f(x)}$) and in L_2. Under simple conditions A_f is also bounded. The ring of all bounded self-adjoint operators A_f with f in L_2 will be denoted by B. According to the Carleman theory the resolution of the identity for A is an integral operator $E_A(\lambda)$

with kernel $\theta(x, y, \lambda)$ when $\lambda<0$; and a similar result holds for $E-E_\Lambda(\lambda)$ when $\lambda>0$. The object of the present paper is to study this kernel in greater detail. It is shown that $\theta(x, y, \lambda)=\theta(xy^{-1}, \lambda)$, where $\theta(x, \lambda)$ is a continuous positive definite function of x (and hence necessarily Hermitian). When f is in L_1 it is shown that $\theta(x, \lambda)$ has a derivative with respect to $\theta(e, \lambda)$ in the sense that $\theta(x, \lambda)=\int_{-\infty}^{\lambda}\varphi(x, \mu)d\theta(e, \lambda)$ for $\lambda<0$; here the function $\varphi(x, \lambda)$ is for almost every λ a continuous positive definite function of x. For $\lambda>0$ similar results hold. If now C is a commutative subring of B and $A=A_f$ a particular member of C, then there is a bounded self-adjoint operator R of which all members of C are functions; in particular, $A=F(R)$. Though R may not be in B, its spectral analysis is shown to resemble that of members of B. It has a spectral kernel of similar nature: if $E_R(\lambda)$ is its resolution of the identity, then for $\lambda<0$ the latter operator is an integral operator with kernel $\tau(xy^{-1}, \lambda)$, where $\tau(x, \lambda)$ is a continuous positive definite function of x; and a similar result holds for $\lambda>0$. In the case where f is in L_1, the function $\tau(x, \lambda)$ has a derivative with respect to $\tau(e, \lambda)$ almost everywhere in the set where $F(\lambda)\neq 0$; this derivative $\omega(x, \lambda)$ is a continuous positive definite function of x for almost all λ. The operator $E_R(\lambda)A_f$ has for $\lambda<0$ the kernel $\int_{-\infty}^{\lambda}F(\mu)\omega(xy^{-1}, \mu)d\tau(e, \mu)$. When C is a maximal commutative subring of B more can be established. When G is Abelian, B is commutative and C has to be taken equal to B. In this case $\omega(x, \lambda)$ is a character of G for almost every λ; and the formula $F(\lambda)=\int_G f(y^{-1})\omega(y, \lambda)dy$ is verified. In the general case of a maximal C, it can be shown that $\omega(x, \lambda)$ is an extremal continuous positive definite function of x, in the sense that, if $\omega_1(x, \lambda)$ and $\omega_2(x, \lambda)$ are continuous positive definite and $\omega(x, \lambda)=\omega_1(x, \lambda)+\omega_2(x, \lambda)$ for almost all λ, then $\omega_1(x, \lambda)=\Phi_1(\lambda)\omega(x, \lambda)$ and $\omega_2(x, \lambda)=\Phi_2(\lambda)\omega(x, \lambda)$ almost everywhere. *M. H. Stone* (Chicago, Ill.).

Godement, Roger. Analyse harmonique dans les groupes centraux. I. Fonctions centrales et caractères. C. R. Acad Sci. Paris **225**, 19–21 (1947).

The author investigates harmonic analysis on a special kind of group similar to an extension of an Abelian group by a compact group. A "central" group G is a locally compact group in which the group of inner automorphisms is a totally bounded subgroup of a topologized group of all continuous automorphisms of G. A "central" function on G is one which is invariant under inner automorphisms almost everywhere relative to Haar measure. A "character" of G is an extreme point of the class of continuous central positive definite functions on G which are less than or equal to 1 at the identity e. Among other results, the author states that, if G is a central group, then: (a) if $x\neq e$, there is an element x^* of the collection G^* of all characters of G such that $x^*(x)\neq x^*(e)$; (b) G^* is locally compact in the weak topology of linear functionals on $L_1(G)$; (c) a generalization of the Herglotz-Bochner-Weil theorem to central positive definite functions on G is valid. *I. E. Segal.*

Bates, Grace E. Free loops and nets and their generalizations. Amer. J. Math. **69**, 499–550 (1947).

Le terme de "half-net" désigne un ensemble formé de points et de trois sortes de lignes tel que (1) par chaque point il passe au plus une ligne de chaque espèce, (2) deux lignes se rencontrent au plus en un point. Un "net" satisfait aux conditions plus précises (1) par chaque point passe exactement une ligne de chaque espèce, (2) deux lignes d'espèces différentes se rencontrent exactement une fois. L'auteur étudie les extensions, les extensions libres (montre que leurs sous "nets" sont également libres), les sommes libres. Elle applique ceci à la notion correspondante de "half loop," ensemble muni d'une opération non toujours définie telle que $a+x=a+y$ entraîne $x=y$. "Half loop" libre, somme libre. Indications sur la théorie des "loops."

J. Kuntzmann (Grenoble).

Mituhisa Takasaki. Abstraction of symmetric transformations. Tôhoku Math. J. **49**, 145–207 (1943). (Japanese)

The author develops a theory of an algebraic system which he calls a "kei" (pronounced "kay"), i.e. a set of elements $a, b, c, \cdots$ on which there is defined a multiplication $c=ab$ with the following properties: (1) $aa=a$, (2) $(ab)b=a$, (3) $(ab)c=(ac)(bc)$. Neither commutative nor associative law is assumed. Neither the existence nor the uniqueness (nor even the finiteness) of the solution of the equation $ax=b$ follows from (1), (2), (3), and this fact makes it more difficult to investigate the structure of keis than that of groups.

An example of a kei is given by the set of all integers if we put $ab=2b-a$. More generally, any additively written Abelian group is a kei if ab is defined in the same way. Thus, the set S_0 of all points on the plane is a kei if ab means the point which is symmetric to a with respect to b. Similarly, the set S_1 of all straight lines on the plane is a kei if ab means the straight line which is symmetric to a with respect to b. It is to be observed that the set $S_0 \cup S_1$ is again a kei if ab is defined similarly. There are many other examples of keis with different structure. For example, the set of all pairs of integers at least one of which is even is a subkei of S_0 while it is not a subgroup of the group S_0.

The author discusses special keis which satisfy the following additional conditions (4) $a(bc)=(ab)(ac)$, (5) $(ab)(cd)=(ac)(bd)$, (6) $(ab)c=(ac)b$, of which each is stronger than the preceding. The whole theory is an extension and continuation of the work of G. Thomsen [Grundlagen der Elementargeometrie in gruppenalgebraischer Behandlung, Hamburger Math. Einzelschr., no. 15, Teubner, Leipzig, 1933] and, as the author remarks, is only in the beginning. There are more definitions than theorems, and the results obtained are rather fragmentary. *S. Kakutani* (Osaka).

NUMBER THEORY

Ratib, I. Une proposition sur les résidus minimums. Proc. Math. Phys. Soc. Egypt 3 (1946), no. 2, 9–12 (1947).

Let k and m be relatively prime integers, and r_μ the numerical value of the absolutely smallest residue of $\mu k \pmod m$ for $\mu=1, 2, \cdots, [m/2]$. Denote by $B(k, m)$ the set of those numbers μ for which $r_\mu<r_\nu$ for each $\nu<\mu$, and by $A(k, m)$ the set of the corresponding numbers r_μ. The author proves the following theorem. If $qk\equiv 1 \pmod m$, then $A(k, m)=B(q, m)$ and $B(k, m)=A(q, m)$. *A. Brauer.*

Schupfer, Francesco. Su due proposizioni di teoria dei numeri. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 246–251 (1946).

(i) It is shown, by use of the quadratic reciprocity law, that a prime number n is a quadratic residue of all and only those primes of the form $p=4nk\pm(2m+1)^2$, where $2m+1$ ranges over the odd integers numerically less than n; if $n=4g+1$, this formula can be replaced by either of $p=2nh+(2m+1)^2$ or $p=2nh-(2m+1)^2$. (ii) If p is a prime num-

ber of the form $6k+1$ with $k>2$, then the congruence $x^3+y^3\equiv z^3 \pmod p$ has solutions with xyz prime to p. The proof considers the residue classes (a) the cubic residues $u_1, \cdots, u_r$; (b) $u_1v, \cdots, u_rv$; (c) $u_1v^2, \cdots, u_rv^2$, where $r=(p-1)/3$ and $1, v, v^2$ are the roots of the congruence $v^3\equiv 1 \pmod p$; and shows by examination of the numbers 1, 2, 3, 7, 13 and $p-1$ that there are three numbers in one residue class such that one is the sum of the other two. Thus, if 2 is in (a), take 2, $p-1$, $p+1$; if 2 is in (b) and 3 in (a), take 1, 8, 9; $\cdots$; if 2 is in (b) and 3 in (c), take 1, 6, 7, or 8, 28, 36, or 6, 8, 14, according as 7 is in (a), (b) or (c). *G. Pall* (Chicago, Ill.).

Schwarz, Štefan. Contribution à la réductibilité des congruences binomiques. Časopis Pěst. Mat. Fys. 71, 21–31 (1946). (Czech. French summary)

The author proves the following theorem. Let p be a prime, $(n, p)=(a, p)=1$, $d_k=(p^k-1, n)$, and σ_k the number of irreducible factors of degree k of the binomial congruence $x^n-a\equiv 0 \pmod p$; then, for $1\leqq k\leqq n$, $\sum_{i|k}i\sigma_i\equiv 0$ or $d_k \pmod p$ according to whether $a^{(p^k-1)/d_k}\not\equiv 1$ or $\equiv 1 \pmod p$. For $n<p$ the σ_k can be determined from these relations. The following applications are given: (1) x^n-a is irreducible mod p if and only if $a^{(p^k-1)/d_k}\not\equiv 1$ for $k<n$, $\equiv 1$ for $k=n$; (2) x^4+1 is reducible mod p for every prime; if $p=8m+3$, 5, 7 there are two quadratic factors, if $p=8m+1$ there are four linear factors; (3) let q be a prime, $q<p$, and let p belong to the exponent l mod q; then the congruence $x^q-a\equiv 0 \pmod p$ is irreducible if $l=1$ and $a^{(p-1)/q}\not\equiv 1 \pmod p$, has q linear factors if $l=1$ and $a^{(p-1)/q}\equiv 1 \pmod p$, and has one linear and $(q-1)/l$ factors of degree l if $l>1$. *F. A. Behrend.*

Vandiver, H. S. Limits for the number of solutions of certain general types of equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. 33, 236–242 (1947).

The author considers the number of solutions x^m, y^m of the equation $1+ax^m=by^m$, where a, b, x, y are in a finite field $F(p^t)$, with m a factor of p^t-1, and $abxy\neq 0$. If i and j are the indices, with respect to some primitive root of the field, of the coefficients a, b, respectively, let (i, j) be the number of solutions x^m, y^m of the above equation. The purpose of this paper is to evaluate the expression $A_{hk}=\sum_{i,j=0}^{m-1}(i, j)(i+h, j+k)$ explicitly. Thus it is found that in case $h-k$, h, k are all incongruent to 0 (mod m), then $A_{hk}=c^2$, where $c=(p^t-1)/m$. *H. W. Brinkmann.*

Țino, O. N. Sur la réduction de l'équation indéterminée du second degré $x^2-\rho y^2=l$ ($\rho>0$) aux équations $x^2-\rho y^2=\pm 1$ et $\alpha x^2-\beta y^2=\pm 1$. Bull. Sci. École Polytech. Timișoara 10, 43–68 (1941).

In this paper the author outlines a method for reducing the indeterminate equation $x^2-\rho y^2=l$, where x, y, ρ, l denote integers with $\rho>0$ and $|l|>\sqrt{\rho}$, to an equation of the same form, with $|l|<\sqrt{\rho}$, or to one of the forms $u^2-\rho v^2=\pm 1$, $\alpha u^2-\beta v^2=\pm 1$, where α, β, u, v are integers and $\alpha\beta=\rho$. Noting that when the given equation is possible in integers, a transformation employed by Lagrange always permits us to obtain one or more solutions, the author discusses various cases in detail. He first considers $x^2-\rho y^2=l$, with $|l|>\sqrt{\rho}$, l a prime, $\rho>0$ and prime to l. It is shown that in this case the equation is resolved into one of $u^2-\rho v^2=-\epsilon$, $k_1^2u^2-\rho_1v^2=-\epsilon$, $u^2-\rho v^2=-2\epsilon$, $k_1u^2-\rho_1v^2=-2\epsilon$, where $\rho=k_1\rho_1$ and ϵ is a definite one of -1, $+1$. Since the general solutions of the first two, and of the last two for $2<\sqrt{\rho}$, have been fully treated by Lagrange [Dickson, History of the Theory of Numbers, v. 2, Washington, 1920, pp. 358–362], it remains only to consider the last two for $\rho\leqq 4$. When integer solutions are possible, it is proved that for $\rho\leqq 4$ these equations are reducible to the form $u^2-\rho v^2=-\epsilon$. The author finally applies to his method to $x^2-ly^2=\rho$ with $|l|>\sqrt{\rho}$, where ρ (not a square) is greater than zero and prime to l, and l is the product of two primes l_1, l_2, both of which are different from 2. In all cases the analysis is similar to that in the first case. He remarks that the treatment in the cases given can be applied without essential modification to the general case with $|l|>\sqrt{\rho}$, where $\rho>0$ is not a square and is prime to l, and l is the product of n factors $l_1, l_2, \cdots, l_n$. *W. H. Gage.*

Moessner, Alfred, und Gloden, Albert. Einige zahlentheoretische Untersuchungen und Resultate. Bull. Sci. École Polytech. Timișoara 11, 196–219 (1944).

Examples of the determination of numerical solutions of several Diophantine systems such as

$$(1)\qquad x_1, \cdots, x_k \overset{n}{=} y_1, \cdots, y_k$$

or $x_1, \cdots, x_k \overset{n}{=} y_1, \cdots, y_k \overset{n}{=} z_1, \cdots, z_k \overset{n}{=} u_1, \cdots, u_k$, $n=2, 4$, and of more complicated systems such as $u_1, \cdots, u_6 \overset{4}{=} v_1, \cdots, v_6$, $u_4\cdot u_5\cdot u_6 \overset{n}{=} v_4\cdot v_5\cdot v_6$, $u_1\cdot u_2\cdot u_3 \overset{n}{=} v_1\cdot v_2\cdot v_3$, $n=1, 4$. Two parametric solutions of (1) for $k=6$, $n=1, 3, 5, 7$. Some numerical solutions of the Tarry-Escott problem [Dorwart and Brown, Amer. Math. Monthly 44, 613–633 (1937)]. Parametric solutions of $y^2-z^2=u^2$, $x^2-z^2=v^2$, $x^2-y^2=w^2$. See for theorems, A. Gloden, Mehrgradige Gleichungen [Noordhoff, Groningen, 1944; these Rev. 8, 441].

N. G. W. H. Beeger (Amsterdam).

Palamà, G. Generalizzazione di due teoremi sulle uguaglianze multigrade, su delle trasformazioni di esse e sulle multigrade a catena. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 95–120 (1947).

The main theorem is as follows. From

$$(1)\qquad a_1, \cdots, a_r \overset{n}{=} b_1, \cdots, b_r, \qquad r>n,$$

follows: if n is even,

$$a_1, \cdots, a_r, k-a_1, \cdots, k-a_r \overset{n+2}{=} b_1, \cdots, b_r, k-b_1, \cdots, k-b_r;$$

if n is odd,

$$a_1, \cdots, a_r, k-b_1, \cdots, k-b_r \overset{n+2}{=} b_1, \cdots, b_r, k-a_1, \cdots, k-a_r,$$

where

$$k=\frac{2(T_{n+2}-S_{n+2})}{(n+2)(T_{n+1}-S_{n+1})}, \quad T_n=\sum_i b_i^n, \quad S_n=\sum a_i^n$$

[for notation see these Rev. 8, 441 (review of Gloden's "Mehrgradige Gleichungen")]. Proof by the binomial theorem. The theorem is proved to be a generalization of two known theorems. Further theorems on transformation of multigrade equalities, for example: from (1) follows a_1+a_2, $a_1+a_3, \cdots, a_1+a_r, a_2+a_3, \cdots, a_2+a_r, \cdots, a_{r-1}+a_r \overset{n}{=} b_1+b_2$, etc. Construction of chains of equalities (1), for example: from (1) follows $\lambda_i a_1+\alpha_i, \lambda_i a_2+\alpha_i, \cdots, \lambda_i a_r+\alpha_i \overset{n}{=} \lambda_i b_1+\alpha_i$, etc., $i=1, \cdots, m$. Now interchanging the members of 0, then of 1, then of 2, $\cdots$, then of m equalities of this system we get 2^m systems. We may form a chain whose members are the 2^m sums of the members of each system. Each member has nr terms. *N. G. W. H. Beeger* (Amsterdam).

Palamà, G. Metodi per avere soluzioni parametriche della $a_1, \cdots, a_p \overset{2,4}{=} b_1, \cdots, b_p$, **nei casi** $p=3$, $p=4$. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 48–64 (1947).

Methods for deriving parametric solutions of

$$a_1, a_2, a_3 \overset{2,4}{=} b_1, b_2, b_3$$

and of $a_1, a_2, a_3, a_4 \overset{2,4}{=} b_1, b_2, b_3, b_4$. Relations between them.

N. G. W. H. Beeger.

Palamà, G. Quelques théorèmes sur les multigrades. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **16**, 98–103 (1946).

This paper is concerned with the Tarry-Escott problem of finding two sets of integers with equal sums of like powers. The solutions are special in that each set is such that its members can be grouped in pairs having a constant sum. Given two such sets the author obtains two others as linear combinations of the members of the first sets. Thus two-parameter solutions can be set down. For example, from the fact that the two sets

$$1,\ 6,\ 7,\ 17,\ 18,\ 23;$$
$$2,\ 3,\ 11,\ 13,\ 21,\ 22$$

have equal sums of kth powers, for $k=0, 1, \cdots, 5$, it is deduced that the two sets

$$p+2q,\ 6p+3q,\ \cdots,\ 23p+22q;$$
$$2p+q,\ 3p+6q,\ \cdots,\ 22p+23q$$

have the same property. Applications are given to the problem of 3 or more sets with equal sums of like powers.

D. H. Lehmer (Berkeley, Calif.).

***Hoheisel, G. Über Diophantische Gleichungen.** Ber. Math.-Tagung Tübingen 1946, pp. 81–84 (1947).

Let $F(x)=F(x_1, \cdots, x_n)$ be a homogeneous polynomial with integral coefficients, K an algebraic number field of degree n. Say that a rational integer h is representable by F if there exist integers $x_1, \cdots, x_n$ in K such that $F(x)=h$. Under eight hypotheses (which are not reproduced here) imposed on F, it is shown that, for all but a finite number of primes p, if gp is representable for some g it is representable with g less than a constant dependent on n.

I. Niven (Eugene, Ore.).

Niven, Ivan. A simple proof that π is irrational. Bull. Amer. Math. Soc. **53**, 509 (1947).

Of all known proofs for the irrationality of π the author's seems to be the simplest. It proceeds as follows. Let $\pi=a/b$ be rational, where a and b are positive integers. Put $f(x)=b^n x^n(\pi-x)^n/n!$ and $F(x)=f(x)-f^{(2)}(x)+f^{(4)}(x)-\cdots+(-1)^n f^{(2n)}(x)$, the positive integer n being specified later. Then the relation (1) $\int_0^\pi f(x)\sin x\,dx=F(0)+F(\pi)$ can be proved. From this we deduce a contradiction; for the left-hand side represents a positive number, tending to zero if n increases indefinitely, and the right-hand side is an integer, as is easily shown.

J. Popken (Utrecht).

Khintchine, A. Deux théorèmes liés au problème de Tchebycheff. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 105–110 (1947). (Russian. French summary)

The author proves the following two theorems. (1) Let θ be irrational, and let β and C be positive constants. Then there exists a real number α such that $\theta x-y-\alpha\neq 0$ for all integers x, y, and that furthermore

(a) $$|\theta x-y-\alpha|<1/t,\quad |x|<Ct^\beta$$

is soluble in integers x, y for every real $t\geqq 1$ if and only if there is a positive constant γ such that $|q\theta-p|<\gamma q^{-1/\beta}$ for an infinity of pairs of integers $p, q>0$. (2) Denote by ξ_q the distance from ξ to the nearest fraction of denominator q; thus $0\leqq\xi_q\leqq\frac{1}{2}q^{-1}$. The conditions (a) are soluble in integers x, y for every $t\geqq 1$ if and only if there is a constant Γ such that $\alpha_q<\Gamma\theta_q^{1/(1+\beta)}$ for every integer $q\geqq 1$.

K. Mahler.

Jarník, Vojtěch. On linear inhomogeneous Diophantine approximations. Rozpravy II. Třídy České Akad. **51**, no. 29, 21 pp. (1941). (Czech)

Let θ_{ij}, α_i, β_j $(1\leqq i\leqq r,\ 1\leqq j\leqq s)$ be given real numbers, and let a, b denote integers. Define, for $t\geqq 1$, $\psi_1(t; \alpha_1, \cdots, \alpha_r)$ as the minimum for $0<\max|a_j|\leqq t$ and $1\leqq j\leqq s$ of

$$\max_{1\leqq i\leqq r}|\theta_{i1}a_1+\cdots+\theta_{is}a_s+a_{i+s}+\alpha_i|;$$

$\psi_2(t; \beta_1, \cdots, \beta_s)$ as the minimum for $0<\max|b_{s+i}|\leqq t$ and $1\leqq i\leqq r$ of

$$\max_{1\leqq j\leqq s}|\theta_{1j}b_{s+1}+\cdots+\theta_{rj}b_{s+r}-b_j-\beta_j|,$$

and put $\psi_1(t)=\psi_1(t; 0, \cdots, 0)$, $\psi_2(t)=\psi_2(t; 0, \cdots, 0)$. The function ψ_1' is defined in a similar way to ψ_1 except that the condition $0<\max|a_j|$ is omitted. Further, let $\varphi(t)$, $\sigma(t)$ be any increasing continuous functions of t for $t\geqq 0$, for which $\varphi(0)=\sigma(0)=0$ and such that, for some $\eta>0$, $t^{-\eta}\varphi(t)$ is increasing and tends to ∞ as $t\to\infty$; let $\rho(t)$ be the inverse function to $\varphi(t)$.

The author proves the following results, where the limits are taken as $t\to\infty$. (I) $\limsup t^{s/r}\psi_1(t)\leqq 1$, $\limsup t^{r/s}\psi_2(t)\leqq 1$. (II) If $\liminf t^{r/s}\psi_2(t)>0$, then $\liminf t^{s/r}\psi_1(t)>0$. (III) If $A>0$ and $\limsup\varphi(t)\psi_2(t)>A$, then

$$\liminf\rho(t)\sup\psi_1(t; \alpha_1, \cdots, \alpha_r)\leqq\Delta,$$

where sup denotes the least upper bound for $0\leqq\alpha_i<1$, and $\Delta=\frac{3}{2}(r+s)!(r+s)\max[1, \{(r+s)!(r+s)/\Delta\}^{1/\eta}]$.

(IV) If $\limsup\varphi(t)\psi_2(t)<\infty$ then there exists a system $(\alpha_1, \cdots, \alpha_r)$ such that

$$\liminf\rho(t)\psi_1(t; \alpha_1, \cdots, \alpha_r)\geqq\liminf\rho(t)\psi_1'(t; \alpha_1, \cdots, \alpha_r)>0.$$

(V) If the series (A) $\sum_{x=1}^\infty x^{s-1}\sigma^{-r}(x)$ converges, then, for almost all systems $(\alpha_1, \cdots, \alpha_r)$, (B) $\liminf\sigma(t)\psi_1(t; \alpha_1, \cdots, \alpha_r)=\liminf\sigma(t)\psi_1'(t; \alpha_1, \cdots, \alpha_r)=\infty$. (VI) If $\lim t^{-s/r}\sigma(t)=\infty$, and $\liminf\psi_2(t)t^{r/s}>0$, then (a) if the series (A) converges (B) is satisfied for almost all $(\alpha_1, \cdots, \alpha_r)$, and (b) if (A) diverges then for almost all $(\alpha_1, \cdots, \alpha_r)$

$$\liminf\sigma(t)\psi_1(t; \alpha_1, \cdots, \alpha_r)=\liminf\sigma(t)\psi_1'(t; \alpha_1, \cdots, \alpha_r)=0.$$

The proof of (I), (II) and (III) follows the method of two earlier papers [Mahler, Časopis Pěst. Mat. Fys. **68**, 93–102 (1939); Jarník, ibid., 103–111 (1939)].

R. A. Rankin.

Oppenheim, A. Remark on the minimum of quadratic forms. J. London Math. Soc. **21** (1946), 251–252 (1947).

The author proves again the result of Mordell [same J. **19**, 3–6 (1944); these Rev. **6**, 57] that $\gamma_n^{n-2}\leqq\gamma_{n-1}^{n-1}$, where γ_n is the upper bound of the minima of positive definite quadratic forms in n variables, of determinant 1. He presents the proof in such a way as not to presuppose the existence of γ_n.

H. Davenport (Stanford University, Calif.).

Mahler, K. On the minimum determinant and the circumscribed hexagons of a convex domain. Nederl. Akad. Wetensch., Proc. **50**, 692–703 = Indagationes Math. **9**, 326–337 (1947).

Let K denote a plane convex domain, symmetric with respect to the origin, $V(K)$ its area and $\Delta(K)$ the lower

bound of determinants of lattices whose only point interior to K is the origin. Such a lattice with determinant equal to $\Delta(K)$ is known as a critical lattice. The author shows easily that a hexagon H, symmetric with respect to the origin, exists, enclosed by three pairs of parallel tangents to K, so that $\Delta(H)=\Delta(K)$. As H has a single critical lattice, namely the lattice containing the midpoints of the sides of H, a determination of a circumscribed hexagon H, with $\Delta(H)$ a minimum, provides a method of determining $\Delta(K)$ as well as the lattice which defines it. By use of these hexagons a tangent is shown to exist at every point of K which is a point of a lattice critical to K. The method is applied mainly to $2n$-sided polygons Π_n symmetric with respect to the origin. Let Q be the lower bound of $V(K)/\Delta(K)$, where K runs through all plane convex domains, and Q_n the lower bound of the same quantities, where K runs through all polygons Π_n. The existence of a polygon Π_n with $V(\Pi_n)/\Delta(\Pi_n)=Q_n$ is proved and if $n=4$ such a polygon is shown to be a regular octagon. Within a regular octagon Π_4 an irreducible convex domain K is found with $\Delta(\Pi_4)=\Delta(K)$. The quantity $V(K)/\Delta(K)$ is then computed for this domain, thus giving an upper bound for the constant Q. *D. Derry* (Vancouver, B. C.).

Davenport, H., and Rogers, C. A. Hlawka's theorem in the geometry of numbers. Duke Math. J. **14**, 367–375 (1947).

The authors give some improvements of the results of K. Mahler [same J. **13**, 611–621 (1946); these Rev. **8**, 444] and C. A. Rogers [to appear in Ann. of Math.]. (I) Let c_n denote the upper bound of V/Δ for all convex bodies K in n dimensions, symmetrical about the origin O, where Δ is the determinant of any lattice with no point (except O) in K and V the volume of K. Then, for $n\geqq 3$, $c_n\geqq 2n^{-1}(c_{n-1}^{n/(n-1)}-1)/(c_{n-1}^{1/(n-1)}-1)$ and $\lim\inf_{n\to\infty} c_n\geqq c$, where $\log c=2(1-1/c)$, $c>1$ (the numerical value of c is $4.921\cdots$). (II) Let Q be any positive definite quadratic form in n variables of determinant 1 and let $M(Q)$ denote the minimum value of Q for integral values of the variables, not all zero. Let γ_n be the least number such that $M(Q)\leqq\gamma_n$ for all such Q and let $\Theta=0.596\cdots$ denote the minimum, for $0<\eta<\pi$, of $2(\log\pi/\eta)^{-1}\sum_{i=1}^{\infty}e^{-\eta i^2}$. Then $\lim\inf_{n\to\infty} n^{-1}T(n)\gamma_n^{n/2}\geqq 1/\Theta$, where $T(n)=\pi^{n/2}/\Gamma(1+n/2)$. *V. Knichal* (Prague).

*Ricci, Giovanni. **Problemi secolari e risposte recenti nel campo dell'aritmetica.** Atti Convegno Mat. Roma 1942, pp. 91–131 (1945).

This lecture discusses the estimation of exponential sums, the distribution of $f(x)$ (mod 1), Waring's problem and Goldbach's problem.

Vinogradow, I. M. A general law of the theory of primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 471–472 (1947).

In a previous note with a similar title [same C. R. (N.S.) **51**, 491–492 (1946); these Rev. **8**, 6], the author obtained an estimate for $\sum_{k=1}^{K}|\sum\exp(2\pi ikf(p))|$, where K is a positive constant, the inner sum is over all primes $p\leqq P$, and $f(p)$ is a polynomial with real coefficients. He then used the estimate to establish an asymptotic formula for the number T of fractional parts $\{f(p)\}$ of the polynomial satisfying $0\leqq\{f(p)\}\leqq\sigma$, $p\leqq P$. In this note the results are extended to the case where $f(p)$ is no longer a polynomial, but a function which may be approximated by a polynomial in a certain sense explained in the note. Two examples are given. One is the function a/p^m to which the stated results apply, and the other is the function $a\log x$, to which the results do not apply. No proofs are given. *R. D. James* (Vancouver, B. C.).

Tchudakoff, N. On Goldbach-Vinogradov's theorem. Ann. of Math. (2) **48**, 515–545 (1947).

The author proves Vinogradov's theorem that every large odd integer is representable as a sum of three odd primes. The proof is very similar to the latest proof given by Linnik [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 3–8 (1946); Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 35–46 (1946); these Rev. **8**, 317, 11]. *H. A. Heilbronn* (Bristol).

*Ullrich, Egon. **Zum Zwillingssatz von Viggo Brun.** Ber. Math.-Tagung Tübingen 1946, pp. 139–143 (1947).

The author states that Brun's theorem on the convergence of $\sum p^{-1}$, summed over the primes p for which $p+2$ is also a prime, can be extended to the case when $p+2$ is replaced by $p+2q$ for any fixed q. He also makes some observations on the probable distribution of such primes. *H. Davenport* (Stanford University, Calif.).

Ramaswami, V. On the divisors of a multiplicative set. J. Benares Hindu Univ. **7**, part 1, 177–178 (1943).

Let $a_1, a_2, \cdots$ be a sequence of real numbers with the properties that (i) $a_1=1$, (ii) $a_{n+1}>a_n$, (iii) to every pair of positive integers m and n there corresponds a third integer p such that $a_p=a_ma_n$ and (iv) there exists a constant c_1 such that the number of a_n not exceeding x is $c_1x+O(1)$. By generalizing the standard proofs for the case where $a_n=n$, it is shown that properties (i), (ii) and (iv) imply the existence of an "Euler-Mascheroni constant" c_2, that is,

$$\sum_{a_n\leqq x}1/a_n=c_1\log x+c_2+O(1/x)$$

and that properties (i)–(iv) imply the analogue of Dirichlet's formula,

$$\sum_{a_n\leqq x}d(a_n)=c_1x\log x+(2c_1c_2-c_1^2)x+O(x^{\frac{1}{2}}),$$

where $d(a_n)$ is the number of ordered pairs (a_r, a_s) satisfying $a_n=a_ra_s$. *P. Hartman* (Baltimore, Md.).

Bellman, Richard. A note on relatively prime sequences. Bull. Amer. Math. Soc. **53**, 778–779 (1947).

The author proves the following theorem. Let $f(x)$ be a polynomial with integral coefficients. Put $f_1(x)=f(x)$, $f_{n+1}(x)=f(f_n(x))$. Assume that $f_n(0)=f(0)\neq 0$, and that whenever $(x, f(0))=1$ we also have $(f(x), f(0))=1$. Then if x is any integer for which $(x, f(0))=1$, no two of the numbers $x, f_1(x), \cdots, f_n(x), \cdots$ have a common factor greater than 1. This result generalizes the well-known theorem that all the numbers $2^{2^n}+1$ are pairwise relatively prime. *P. Erdös* (Syracuse, N. Y.).

Bellman, Richard. The Dirichlet divisor problem. Duke Math. J. **14**, 411–417 (1947).

It is proved that, if $d(n)$ denotes the number of divisors of n, and $D(x)=\sum_{n<x}d(n)=x\log x+(2C-1)x+\Delta(x)$, then

$$\int_1^T\{\Delta(x)\}^2x^{-1}dx\sim c_1\log T,\qquad T\to\infty,$$

where c_1 is a positive constant. A corresponding result is obtained in the theory of Euler's ϕ-function. As the author

points out, more general problems of this kind had been considered by A. Walfisz [J. Reine Angew. Math. **169**, 111–130 (1933)] and S. Chowla [Math. Z. **35**, 279–299 (1932)]. The method used here, depending on the Riemann zeta-function, is, however, different from that of Walfisz and Chowla. *E. C. Titchmarsh* (Oxford).

Nasir, A. R. Residue of $\sigma_k(n)$ modulo 2. Amer. Math. Monthly **54**, 274–275 (1947).

Let $\sigma_k(n)$ denote the sum of the kth powers of the positive divisors of the positive integer n and let $n=2^a p_1^{b_1}\cdots p_s^{b_s}$, where $p_1, \cdots, p_s$ are different odd primes. The author shows that $\sigma_k(n)\equiv 1 \pmod 2$ if and only if $b_1, \cdots, b_s$ are all even. *R. A. Rankin* (Cambridge, England).

Lehmer, D. H. The vanishing of Ramanujan's function $\tau(n)$. Duke Math. J. **14**, 429–433 (1947).

The author shows that $\tau(n)\neq 0$ for all n less than the prime 3316799. It is first shown, by an argument which makes use of some of the elementary properties of algebraic integers, that n_0, the least value of n for which $\tau(n)=0$, must be prime. By means of known congruences connecting $\tau(n)$ with the function $\sigma_k(n)$ (the sum of the kth powers of the divisors of n) for various values of k and for the moduli 32, 3, 25, 7 and 691, it is deduced that if $\tau(p)=0$, where p is a prime, then p must be congruent to one of three residue classes modulo $2^5\cdot 3\cdot 5^2\cdot 7\cdot 691=11608800$. By applying a rather complicated theorem due to Wilton [Proc. London Math. Soc. (2) **31**, 1–10 (1930)] on the residue of $\tau(n)$ modulo 23, the number of possible values of p less than $2\cdot 10^7$ is reduced to one, namely the prime 3316799. Whether $\tau(3316799)$ is zero or not is unknown. A sketch of an elementary proof of Wilton's theorem is included. *R. A. Rankin* (Cambridge, England).

Bambah, R. P., and Chowla, S. A note on Ramanujan's function $\tau(n)$. Quart. J. Math., Oxford Ser. **18**, 122–123 (1947).

It is shown that $\tau(n)$ is divisible by $801007200=2^5\cdot 3^3\cdot 5^2\cdot 7\cdot 23\cdot 691$ for almost all n. An extension of this result to 3460351104000 would appear to be possible. *D. H. Lehmer* (Berkeley, Calif.).

Erdös, P. Some asymptotic formulas for multiplicative functions. Bull. Amer. Math. Soc. **53**, 536–544 (1947).

Let $f(n)>0$ be a multiplicative function satisfying $f(p^k)=f(p)$ for every prime p and every integer $k\geqq 1$. The author proves that, if the two series $\sum(f(p)-1)/p$ and $\sum(f(p)-1)^2/p$ converge, then $f(n)$ possesses a mean value; in fact, $\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} f(m)=\prod(1+(f(p)-1)/p)\neq 0$. The methods are "elementary" and depend on the fact that the convergence of the two series implies that the functions $f_k(n)=\prod_{p^a\| n} f(p^a)$ tend, as $k\to\infty$, in relative measure to $f(n)$ [cf. P. Erdös and A. Wintner, Amer. J. Math. **61**, 713–721 (1939); these Rev. **1**, 40, and the correct half of theorem 13 in P. Erdös, Bull. Amer. Math. Soc. **52**, 527–537 (1946); these Rev. **7**, 507]. The author also obtains asymptotic formulae for sums of the form

$$\sum_{n_1+\cdots+n_j=n} f^1(n_1)f^2(n_2)\cdots f^j(n_j),$$

$$\sum_{m=1}^{n} f^1(m+k_1)f^2(m+k_2)\cdots f^j(m+k_j),$$

where $f^1(n), f^2(n), \cdots, f^j(n)$ are multiplicative functions. [There are a number of misprints and obscure statements; e.g., the "remark" on p. 536 is false; the conditions of theorem 1 are not clear; footnote 1 seems unjustified.] *P. Hartman* (Baltimore, Md.).

Erdös, Paul. Some remarks and corrections to one of my papers. Bull. Amer. Math. Soc. **53**, 761–763 (1947).

Let $g(n)$ be a nonnegative multiplicative function and let $(g(n)-1)'$ denote either 1 or $g(n)-1$ according as $|g(n)-1|$ does or does not exceed 1. The author proves that if the two series $\sum(g(p)-1)'/p$, $\sum((g(p)-1)')^2/p$, taken over all primes p, converge, then $g(n)$ possesses an asymptotic distribution function $G(x)$; the converse is true whenever $G(x)$ is not the distribution function which is 0 or 1 according as $x<0$ or $x\geqq 1$. An example shows that this last proviso cannot be omitted. This is a corrected version of a theorem occurring in an earlier paper of the author [same Bull. **52**, 527–537 (1946); these Rev. **7**, 507]. Several minor misprints occurring in the earlier paper are also corrected. *P. Hartman* (Baltimore, Md.).

Kuroda, Sigekatu. Über den Dirichletschen Körper. J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **4**, 383–406 (1943).

Ce travail, à l'exception de ses §§ 5 et 7, est consacré à une modification de la démonstration arithmétique, donnée par Hilbert [Math. Ann. **45**, 309–340 (1894)], du théorème, démontré analytiquement par Dirichlet, que le nombre de classes d'idéaux du composé des corps quadratiques $R(\sqrt{m})$ et $k=R(i)$ (où R est le corps rationnel et où i est l'unité imaginaire) est le produit ou le demiproduit de ceux des $R(\sqrt{m})$ et $R(\sqrt{(-m)})$. L'auteur ne modifie la démonstration de Hilbert que dans sa partie finale, où il remplace l'emploi des "genres de la sorte principale" (Geschlechter der Hauptart) par l'application de la loi de translation de la théorie des corps de classes. En plus, il met ce résultat sous une forme plus simple, en considérant, au lieu des classes d'idéaux ordinaires, les classes d'idéaux modulo (multiplicatif) $(1+i)^3$. Le nombre de telles classes de $R(i, \sqrt{m})$ est toujours le produit de ceux des $R(\sqrt{m})$ et $R(\sqrt{(-m)})$.

Au § 5 il est démontré que chaque classe d'idéaux d'une extension quadratique K de $k=R(i)$ contient un idéal dont la norme par rapport à k ne dépasse pas, en valeur absolue, $(|\delta|/2)^{\frac{1}{2}}$, où δ est le discriminant de K/k. Le § 7 contient des remarques sur le groupe des unités du composé de deux corps quadratiques. *M. Krasner* (Paris).

Kloosterman, H. D. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I. Ann. of Math. (2) **47**, 317–375 (1946).

The object of this paper is to use the transformation theory of general ϑ-functions for the study of the matric representations and the characters of the binary modulary congruence group modulo p^λ. [I suggest and shall use in the sequel the phrase "modulary group" for the German "Modulargruppe" in contradistinction to "modular group" for the German "Modulgruppe."]

For the definition of the ϑ-functions in question we need first an integral square matrix Q of degree n; the quadratic form $x'Qx$ is assumed as definite positive, x being a vector of n components. Let $\Delta>0$ be the determinant of Q. A vector a of n components is called "special" if Qa is divisible by Δ. Let N be a positive integer, τ a complex variable of positive imaginary part and z a complex vector with n components. If g, h, a are special vectors then the following

definition is given:

$$\vartheta_{gh}(z|\tau; a, N)=\sum_{m\equiv a(N\Delta)}(-1)^{h'Q(m-a)/N\Delta^2}$$
$$\times\exp\left\{\frac{\pi i\tau}{N\Delta^2}(m+\tfrac{1}{2}g)'Q(m+\tfrac{1}{2}g)+\frac{2\pi i}{\Delta}(m+\tfrac{1}{2}g)'Qz\right\}.$$

For fixed Q, N, g, h this furnishes for different vectors a exactly $N^n\Delta$ functions which are shown to be linearly independent.

The transformation of the function under $\tau\to\tau+\beta$ with β a rational integer is obvious; for $\tau\to-1/\tau$ the Poisson sum formula is used. Hecke's variant of Hermite's method gives the transformation formula for a general modular substitution through the investigation of certain Gaussian sums; N and Δ are now taken as odd. It turns out that any of the ϑ-functions goes over into a linear aggregate of them, however in general with different g, h. Through a particular choice of g modulo 2 it can be achieved that ϑ_{gg} goes over into an aggregate of ϑ_{gg}'s. In order to get a linear relation with constant coefficients the following quotients are introduced:

$$x_a(z|\tau)=\vartheta_{gg}(z|\tau; a, N)/\vartheta_{gg}^N(z|\tau; 0, 1).$$

These functions are subjected to substitutions of the principal congruence modular group $G(\epsilon_n)$:

$$U=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\pmod{\epsilon_n},$$

where ϵ_n is the greatest elementary divisor of the matrix Q. Here U operates on a function $x_a(z|\tau)$ as follows:

$$x_a(z|\tau)U=(\gamma\tau+\delta)^{\frac{1}{2}n(N-1)}x_a\left(\frac{z}{\gamma\tau+\delta}\middle|\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right).$$

The results about the ϑ_{gg} are used now to show that

$$x_aU=\sum_{b \bmod N\Delta}^{*}A(a, b)x_b,$$

where the asterisk denotes that the summation is to be taken only over "special" vectors. Finally, for a simplification of the transformation matrix, instead of the $x_a(z|\tau)$ the functions

$$u_a(z|\tau)=(-1)^{v'Q(a+c)}x_{a+c}(z|\tau)$$

are introduced, where v and c are certain special vectors defined in terms of Q, N, Δ, g. Then

$$(*)\qquad u_aU=\sum_{b \bmod N\Delta}^{*}C_0(a, b)u_b,$$

where the $C_0(a, b)$ are explicitly determined. It follows that the correspondence (*) $U\to(C_0(a, b))$ is a matrix representation of the group $G(\epsilon_n)$; this representation, denoted by R_0, is $N^n\Delta$-rowed. Since it turns out that $(C_0(a, b))$ is the identity matrix for all substitutions

$$U=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\pmod{N\epsilon_n}$$

of the principal congruence subgroup $G(N\epsilon_n)$ of $G(\epsilon_n)$, (*) actually gives a representation of the quotient group $G(\epsilon_n)/G(N\epsilon_n)$, which is simply isomorphic with the binary modulary group modulo N.

This representation is now studied, especially for $N=p^\lambda$, p odd. This requires first an examination of the sets of conjugate elements in the modulary group modulo p^λ, which are enumerated in a table. The number of complete sets of conjugates is $p^\lambda+4\sum_{\nu=0}^{\lambda-1}p^\nu$. The representation R_0 is found to be the direct sum of Δ identical representations formed by all those u_a for which $a\equiv0 \pmod\Delta$. This reduced representation of N^n rows, called R, is further studied.

Some invariant subspaces of R are found, first by the use of ϑ-relations for N, N_1, N_2, when $N=N_1^2N_2$, and second by considering automorphs modulo N of the quadratic form $x'Qx$. In this way a complete decomposition of R into its irreducible components can be achieved for $n=1$, $N=p^\lambda$. A table shows the 2λ different irreducible characters thus obtained, which are only a part of the total number of all characters. *H. Rademacher* (Philadelphia, Pa.).

Kloosterman, H. D. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. II. Ann. of Math. (2) **47**, 376–447 (1946).

In part I [see the preceding review] a matrix representation R of the group G, the modulary congruence group modulo N, was obtained by means of theta functions of n variables belonging to the quadratic positive definite form $x'Qx$. This representation is studied here for $n=2$, $N=p^\lambda$, p an odd prime. A complete decomposition into irreducible representations is achieved. A number of irreducible characters of G are obtained, which form the complete set for $\lambda=1$ (for this case already found by Frobenius). Tables of the characters are included. The complicated details of the investigation cannot be indicated briefly. The terminology of quadratic fields instead of quadratic forms is used. Of interest is the appearance of certain sums of roots of unity which are generalizations of the "Kloosterman sums"; for these generalizations the author proves here estimates similar to those of Davenport and Salié. *H. Rademacher.*

***Deuring, Max. Teilbarkeitseigenschaften der singulären Moduln der elliptischen Funktionen.** Ber. Math.-Tagung Tübingen 1946, pp. 62–63 (1947).

This is a brief résumé of results published by the author in Comment. Math. Helv. **19**, 74–82 (1946) [these Rev. **8**, 318]. *O. F. G. Schilling* (Chicago, Ill.).

***Petersson, H. Modulformen und Zahlentheorie.** Ber. Math.-Tagung Tübingen 1946, pp. 116–118 (1947).

Brief report of a lecture. *H. S. Zuckerman.*

ANALYSIS

Berwald, L. Verallgemeinerung eines Mittelwertsatzes von J. Favard für positive konkave Funktionen. Acta Math. **79**, 17–37 (1947).

The following theorem, stated here only in its 1-dimensional form, is the main result of the paper. Let $f(x)$ be a continuous nonnegative concave function in the interval $a\leqq x\leqq b$ and let $\phi(y)$ be a continuous strictly increasing function in the infinite interval $y\geqq0$. Then the equation

$$(1)\qquad \zeta^{-1}\int_0^\zeta\phi(y)dy=(b-a)^{-1}\int_a^b\phi(f(x))dx$$

possesses exactly one positive root $\zeta=z$. Furthermore, let $m(y)$ be a bounded nondecreasing function in the interval $0\leqq y\leqq z$ and let the function $\psi(y)$ be defined on this inter-

val by the Stieltjes integral

$$\psi(y)=\int_0^y m(t)d\phi(t).$$

Then

$$(2)\qquad x^{-1}\int_0^x \psi(y)dy \geqq (b-a)^{-1}\int_a^b \psi(f(x))dx.$$

If $\phi(y)=y$, then (1) defines z explicitly and (2) becomes the theorem of Favard [Bull. Sci. Math. (2) **57**, 54–64 (1933)] mentioned in the title. As an application the author shows that the function

$$F(t)=\lim_{s\to t}\left\{[(s+1)/(b-a)]\int_a^b f(x)^s dx\right\}^{1/s},\qquad -1<t\leqq\infty,$$

is nonincreasing. This result includes several known inequalities as special cases. *W. Gustin.*

Ramaswami, V. On the continuity of convex functions. J. Benares Hindu Univ. **7**, part 1, 180–181 (1943).

Let the function f be defined in an open interval I and convex in the sense that $f(\frac{1}{2}(x_1+x_2))\leqq\frac{1}{2}\{f(x_1)+f(x_2)\}$ for all x_1, x_2 in I. The author shows that f is continuous if its superior limits at the two endpoints of I are finite. Reviewer's remark: This theorem is an immediate consequence of the following result: f is continuous if it is bounded from above on some open subinterval of I [Bernstein and Doetsch, Math. Ann. **76**, 514–526 (1915)], or, indeed, merely if it is bounded from above on some subset of I of positive measure [Ostrowski, Jber. Deutsch. Math. Verein. **38**, 54–62 (1929)]. *W. Gustin* (Bloomington, Ind.).

Popoviciu, Tiberiu. Notes sur les généralisations des fonctions convexes d'ordre supérieur. I. Disquisit. Math. Phys. **1**, 35–42 (1940).

For a finite sequence $x_1, x_2, \cdots, x_m$ of distinct points of the set E of definition of a function $f(x)$, divided differences are defined by

$$\Delta_0^i=f(x_i),\quad \Delta_k^i=(\Delta_{k-1}^{i+1}-\Delta_{k-1}^i)/(x_{k+i}-x_i);$$

the Δ_k^i are symmetric in the x-values involved. By definition, $f(x)$ is convex of order n on E provided we have $\Delta_{n+1}^1>0$ for each sequence $x_1, \cdots, x_{n+2}$ of $n+2$ points in E. Any divided difference Δ_{n+1} involving an arbitrary set of $n+2$ points of a finite ordered sequence $S_m: x_1<x_2<\cdots<x_m$, $m\geqq n+2$, can be expressed as a linear combination, with nonnegative coefficients whose sum is 1, of the sequence d_{n+1}^m of divided differences $d_{n+1}^m: \Delta_{n+1}^1, \Delta_{n+1}^2, \cdots, \Delta_{n+1}^{m-n-1}$. Accordingly, f is convex on S_m if and only if each term in d_{n+1}^m is positive. The function $f(x)$ is now defined to be of order $(n|k)$ on S_m provided there are exactly k variations of sign in the sequence d_{n+1}^m; and $f(x)$ is defined to be of order $(n|k)$ on E provided $f(x)$ is of order $(n|k)$ on some finite ordered sequence in E, but is not of order $(n|l)$ for any $l>k$ on any finite ordered sequence in E. In the present note the author develops some elementary properties of functions of order $(n|k)$. Thus it is shown that, if $f(x)$ is of order $(n|k)$, and also of order $(n-i|l)$, on E, then $l\leqq k+i$. *E. F. Beckenbach* (Los Angeles, Calif.).

Popoviciu, Tiberiu. Notes sur les généralisations des fonctions convexes d'ordre supérieur. IV. Disquisit. Math. Phys. **2**, 127–148 (1942).

A function $f(x)$, defined on a set E, is said to be monotone by segments provided E admits a decomposition into consecutive subsets $E_1, E_2, \cdots, E_m$, on each of which $f(x)$ is monotone. The number m has a minimum h, called the characteristic of $f(x)$ relative to E. A function $f(x)$ of characteristic h is of order at least $(0|h-1)$ and at most $(0|2h-2)$. [See the preceding review.] If $f(x)$ is of order $(0|h-1)$, then $f(x)$ is said to be of minimum type. The present paper is devoted to a discussion of functions which are monotone by segments, especially those of minimum type. Theorems concerning existence, uniqueness, continuity, extension and representation are given. *E. F. Beckenbach* (Los Angeles, Calif.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur. IV. Disquisit. Math. Phys. **1**, 163–171 (1940).

[For note III cf. Mathematica, Cluj **16**, 74–86 (1940); these Rev. **2**, 76.] Let $f(x)$ be a continuous nonconcave function of order $2m-1$ in an interval (A, B), let $p(x)$ be nonnegative and summable in a finite interval (a, b), with $\int_a^b p(x)dx>0$, and let $\varphi(x)$ be summable and bounded in (a, b), with $A\leqq\varphi(x)\leqq B$. It is shown that we have

$$\int_a^b p(x)f[\varphi(x)]dx \Big/ \int_a^b p(x)dx \geqq \sum_{j=1}^{m}\mu_j f(y_j),$$

where the y_j are the zeros and the μ_j are the weights of the polynomial P_m of the orthogonal sequence $P_0, P_1, \cdots$, determined by the moments

$$C_k=\int_a^b p(x)\varphi^k(x)dx \Big/ \int_a^b p(x)dx,\quad k=0, 1, \cdots.$$

For $m=1$, the above result becomes the familiar inequality

$$\int_a^b p(x)f[\varphi(x)]dx \Big/ \int_a^b p(x)dx \geqq f\left(\int_a^b p(x)\varphi(x)dx \Big/ \int_a^b p(x)dx\right).$$

A similar result is obtained for nonconcave functions of even order. *E. F. Beckenbach* (Los Angeles, Calif.).

Obrechkoff, N. Sur quelques inégalités nouvelles pour les fonctions de variables réelles. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 213–238 (1946). (Bulgarian. French summary)

This paper contains the proofs of results announced in C. R. Acad. Sci. Paris **222**, 531–533 (1946); these Rev. **7**, 419. Generalizations to functions of several variables are also given. Cf. also C. R. Acad. Sci. Paris **224**, 880–882 (1947); these Rev. **8**, 448, 709. *R. P. Boas, Jr.*

Tagamlizki, J. Funktionen, die auf der reellen achse gewissen Ungleichungen genügen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 239–256 (1946). (Bulgarian. German summary)

The author has outlined a proof that (*) $|f^{(k)}(x)|\leqq Ae^x$ for $k=0, 1, 2, \cdots$ and $x<a$ implies $f(x)=Ce^x$, $|C|\leqq A$ [C. R. Acad. Sci. Paris **223**, 940–942 (1946); these Rev. **8**, 259]. For other proofs see Obrechkoff [C. R. Acad. Sci. Paris **224**, 993–995 (1947); these Rev. **8**, 448] and Boas [C. R. Acad. Sci. Paris **224**, 1683–1685 (1947); these Rev. **8**, 569]. Here the author gives a proof in detail and also shows that the theorem remains true if (*) holds only for $x=x_n$, $x_n\to-\infty$, provided that $f(x)$ is analytic; or if (*) holds only for $k=k_n$, $k_n\to\infty$, if $f(-\infty)=0$. Among the applications are an analogous result in which the conclusion

is $f(x)=C/(x+a)$ and one in which $e^{-x}f^{(k)}(x)$ is replaced by $\Delta^k f(x)$, where $\Delta f(x)=[\varphi(x)f(x)]'/\varphi'(x)$, $\varphi(x)$ being real and of class C^∞ in $a<x<b$, $\varphi(b-)=0$, $\varphi'(x)\neq 0$.

R. P. Boas, Jr. (Providence, R. I.).

Lelong, Pierre. Sur les fonctions indéfiniment dérivables de plusieurs variables dont les laplaciens successifs ont des signes alternés. Duke Math. J. 14, 143–149 (1947).

The following generalization of a theorem of D. V. Widder is proved. Let $f(x_1, \cdots, x_p)=f(x)$ be a function having all derivatives in a certain domain D of p-dimensional space. Let Δ denote the p-dimensional Laplacian, $\Delta^{(n)}$ the corresponding n-times iterated differential operator. If, for all integers n, $(-1)^n\Delta^{(n)}f(x)\geqq 0$ holds in D, then $f(x)$ is analytic in D. Moreover, $f(x)$ is a harmonic function of infinite order in the sense of Aronszajn [Acta Math. 65, 1–156 (1935)]. Various consequences are pointed out.

G. Szegö (Stanford University, Calif.).

Tchakaloff, L. (Čakalov, L.). Sur quelques inégalités entre la moyenne arithmétique et la moyenne géométrique. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 42, 39–44 (1946). (French. Bulgarian summary)

Given numbers $a_n\geqq 0$, let $A_n=(a_1+\cdots+a_n)/n$ and $G_n=(a_1\cdot\cdots\cdot a_n)^{1/n}$. The author verifies that $n(A_n-G_n)$ is a nondecreasing function of n, and that if a_n is nondecreasing then so is $n^2(A_n-G_n)/(n-1)$. *L. C. Young.*

Faguet, M. A generalization of the Hadamard determinant inequality. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 761–764 (1946).

Let $x_1, \cdots, x_m$ be complex n-dimensional vectors. Gram's determinant is the m-rowed determinant of their scalar products; denote by $\Pi(r)$ the product of the $\binom{m}{r}$ determinants similarly formed with selections of r of the given vectors. The author proves very simply that the $\binom{m-1}{r-1}$th root of $\Pi(r)$ decreases as r increases from 1 to m. In the chain of inequalities so obtained, the extreme members give Hadamard's determinant inequality. *L. C. Young.*

Perks, Wilfred. A simple proof of Gauss's inequality. J. Inst. Actuaries Students' Soc. 7, 38–41 (1947).

The author gives an incomplete proof of Gauss's inequality

$$\lambda^2\int_\lambda^\infty \phi(x)dx\leqq\frac{4}{9}\int_0^\infty x^2\phi(x)dx,$$

where $\phi(x)$ is nonincreasing. The method attempted is the same as that found in H. Cramér's Methods of Mathematical Statistics, p. 256, ex. 4 [Princeton University Press, 1946; these Rev. 8, 39]. *R. C. Buck* (Providence, R. I.).

Biernacki, M. Sur les fonctions lentement croissantes. Bull. Sci. École Polytech. Timişoara 12, 146–162 (1946).

A continuous positive increasing function $f(x)$, defined for $0\leqq a<x<1$, is said to be "slowly increasing," abbreviated s.i., if for some $p>1$ (1) $f(x)/f(x^p)<C(1-x)^{-k}$, where C and k are fixed positive numbers. If (1) holds for $k=0$, $f(x)$ is said to be "very slowly increasing," written v.s.i. These classes do not depend upon the choice of p. If $f(x)$ is s.i., $f(x)$ is dominated by $A(1-x)^{-B}\exp k\{\log(1-x)^{-1}\}^2$. The sum and product of s.i. (v.s.i.) functions is again s.i. (v.s.i.). If $f(x)$ is continuous, positive and increasing, and $\log f(x)$ is convex in $\log x$, then $f(x)$ is s.i. if and only if $(1-x)f'(x)/f(x)\log(1-x)^{-1}$ is bounded as $x\to 1$, and is v.s.i. if and only if $(1-x)f'(x)/f(x)$ is bounded. Here $f'(x)$ is the derivative on the right. Turning now to functions $f(z)$ analytic in $|z|<1$, $f(z)$ is said to be s.i. (v.s.i.) if $M(r)$ is s.i. (v.s.i.) as $r\to 1$. Equivalent forms of this are given with $M(r)$ replaced by other growth functions. If the inequality $|f'(z)/f(z)|<C(1-r)^{-1}\log(1-r)^{-1}$ holds for all $r<1$, $f(z)$ is s.i. If $f(z)$ and $g(z)$ are s.i., their sum need not be s.i.; however, $f'(z)$ is s.i. if $f(z)$ is. For the smaller class of v.s.i. functions, the latter of these also fails. The author conjectures that if $f(z)$ is v.s.i., and $F'(z)=f(z)$, then $F(z)$ is v.s.i. Certain familiar classes of functions are v.s.i. In particular, p-valent functions and the first derivative of a univalent function are v.s.i. *R. C. Buck.*

Mirakyan, G. Sur la convergence d'une formule interpolatoire. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 87–90 (1946).

Let $E(x,\mu)=\sum_{k=0}^\infty c_k(\mu)x^k$ be a family of entire functions depending on the positive parameter μ, with nonnegative coefficients $c_k(\mu)$. The following two further assumptions are made. (1) For every $x>0$ we have $\liminf\{E(x,\mu)\}^{1/m}>0$ as $m\to\infty$ and $\mu\to\infty$; (2) $\lim_{\mu\to\infty}E_x'(x,\mu)/\mu E(x,\mu)=1$. It is then shown that a convergent interpolation formula can be derived from $E(x,\mu)$ as follows. Let $f(x)$ be given continuous in $[\rho,r]$ $(0<\rho<r<\infty)$. Let now m (integer) and μ both tend to infinity through such sequences of values that $\liminf(m/\mu)>r$. Suppose that $|f(x)|<M$ in $[\rho,r]$, and let the definition of $f(x)$ be extended to values of $x=k/\mu$ $(k=0,1,\cdots,m)$, falling outside $[\rho,r]$, in such a way that $|f(x)|<M$ and such that $f(x)$ is right-continuous at $x=r$ and left-continuous at $x=\rho$. Then

$$\lim_{\mu\to\infty}\sum_{k=0}^m\frac{c_k(\mu)x^k}{E(x,\mu)}f(k/m)=f(x),$$

uniformly in the range $\rho\leqq x\leqq r$. *I. J. Schoenberg.*

Verblunsky, S. On the initial moments of a bounded function. Proc. Cambridge Philos. Soc. 43, 275–279 (1947).

The author establishes the following theorem I. If $f(x)$ is integrable in $(0,\infty)$ and $0\leqq f\leqq 1$, then there exists a nondecreasing function $\lambda(x)$ such that

$$\int_0^\infty d\lambda=1,\quad \int_0^\infty x\,d\lambda<\infty, \tag{1}$$

$$\exp\int_0^\infty\frac{f(x)}{\zeta-x}dx=\zeta\int_0^\infty\frac{d\lambda}{\zeta-x},\qquad \Im\zeta>0. \tag{2}$$

If $f(x)=0$ for $x>a$, then $\lambda(x)$ is constant for $x>a$. Conversely, if $\lambda(x)$ is nondecreasing in $[0,\infty)$ satisfying (1), then there is a function $f(x)$, integrable in $(0,\infty)$, such that $0\leqq f\leqq 1$, and such that (2) holds.

The significance of this result for the initial moments problem for bounded functions is shown by deriving from it the following theorem II. Given the numbers $s_0, s_1, \cdots, s_{n-1}$, define the numbers $r_0, r_1, \cdots, r_n$ by $\exp(\sum_0^{n-1}s_k\zeta^{-k-1})=\sum_0^n r_k\zeta^{-k}+\cdots$. Let $0<a\leqq\infty$. In order that there be a function $f(x)$ such that $0\leqq f\leqq 1$ and

$$s_k=\int_0^a x^k f(x)dx,\qquad k=0,1,\cdots,n-1,$$

it is necessary and sufficient that there be a nondecreasing function $\lambda(x)$ such that

$$r_k=\int_0^a x^k d\lambda(x),\qquad k=0,1,\cdots,n.$$

If $a<\infty$, theorem II is due to Achyèser and Krein [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 12, 13–35 (1935)]. It is new for $a=\infty$.

I. J. Schoenberg (Philadelphia, Pa.).

Turán, Paul. Sur la théorie des fonctions quasi analytiques. C. R. Acad. Sci. Paris 224, 1750–1752 (1947).

L'auteur établit une propriété de quasi-analyticité de certaines séries d'exponentielles $F(x)=\sum_{\nu=1}^{\infty}a_\nu e^{(h_\nu+ik_\nu)x}$, h_ν réels. La classe $C(a_\nu)$ où les a_ν possèdent la propriété: il existe $\rho>0$ tel que

$$(*)\qquad \limsup_{\omega\to\infty} e^{(2/\rho)\omega\log\omega}\sum_{\nu>\omega}|a_\nu|<\infty,$$

est une classe quasi-analytique au sens suivant: deux fonctions F_1, F_2 de $C(a_\nu)$ telles que pour un x_0 on ait

$$\lim_{h\to+0} e^{h^{-\rho}}\max_{x_0-h\le x\le x_0}|F_1(x)-F_2(x)|<\infty$$

sont égales presque partout.

Quand $h_\nu=m_\nu$ entier, la classe $C(a_\nu)$ de l'auteur est à rapprocher des classes quasi-analytiques de fonctions sommables définies à partir de leurs coefficients de Fourier [cf. S. Mandelbrojt, Séries de Fourier et Classes Quasi Analytiques, Gauthier-Villars, Paris, 1935, p. 707] et d'un résultat de Mandelbrojt qui suppose la suite m_ν d'exposant $\sigma<1$. Par comparaison la classe $C(a_\nu)$ de l'auteur a l'intérêt de faire intervenir les a_ν, ν pouvant prendre toute valeur entière; on remarquera en contre-partie la caractère restrictif de la condition (*).

P. Lelong (Lille).

Calculus

***Widder, David V. Advanced Calculus.** Prentice-Hall, Inc., New York, 1947. xvi+432 pp. $5.00.

The most noticeable feature of this book is the extreme precision in the statement of theorems and their proofs. Unlike most advanced calculus books, this one is entirely satisfactory as regards rigor. Moreover, it is much richer in mathematical content than most advanced calculus books. The chapter headings are: Partial differentiation, Vectors, Differential geometry, Applications of partial differentiation, Stieltjes integral, Multiple integrals, Line and surface integrals, Limits and indeterminate forms, Infinite series, Convergence of improper integrals, The gamma function, Evaluation of definite integrals, Fourier series, The Laplace transform, Applications of the Laplace transform.

N. Levinson (Cambridge, Mass.).

González, Mario O. Some properties of the H_r functions and their application to the elementary theory of differential equations. Revista Soc. Cubana Ci. Fís. Mat. 2, 79–86 (1946). (Spanish)

The author calls $f(x, y, z, \cdots, w)$ a function of type H_r if for certain fixed numbers $m, n, \cdots, q$ and r the relation $f(tx, t^m y, t^n z, \cdots, t^q w)=t^r f(x, y, z, \cdots, w)$ holds identically in $t, x, y, \cdots, w$. Such functions are considered as generalizations of homogeneous functions. The paper contains analogues of Euler's theorem and its converse. The applications deal with problems of separating the variables and finding integrating factors for first order differential equations involving a function of type H_r in two variables.

A. E. Taylor (Cambridge, England).

Alaci, V. Fonctions pseudo-homogènes et une nouvelle classe d'équations différentielles et aux dérivées partielles. Bull. Sci. École Polytech. Timişoara 11, 6–13 (1943).

A pseudo-homogeneous function satisfies

$$f(t^{\alpha_1}x_1, \cdots, t^{\alpha_n}x_n)=t^m f(x_1, \cdots, x_n),$$

where the α_k are algebraic numbers.

Carafa, Mario. Generalizzazione della trasformazione di un integrale curvilineo in superficiale. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 5, 327–342 (1946).

Proof of the formula of Green and Gauss under conditions of continuity which are less restrictive than the usual ones.

I. Opatowski (Ann Arbor, Mich.).

Rădulet, Remus. Zur Theorie der Vektorfelder. Bull. Sci. École Polytech. Timişoara 9, 309–316 (1940).

The author discusses contour integrals of vector quantities.

I. Opatowski (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

Denjoy, Arnaud. Les ensembles finis. C. R. Acad. Sci. Paris 224, 612–615 (1947).

Utilizing just the idea of an ordered set, of the initial element, and of the terminal element, the idea of a finite cardinal number is built up in terms of finite ordinal types. The procedure is intuitive and is untouched by the logistic method.

R. M. Martin (Bryn Mawr, Pa.).

Cuesta, N. Number of types of ordering. Revista Mat. Hisp.-Amer. (4) 7, 3–9 (1947). (Spanish)

Let m be a cardinal greater than or equal to that of the continuum; then each of the numbers of continuous order types, of order types containing no continuous subtypes, of each of three other special classes of order types, is 2^m. The proof is constructive and rests on a construction of 2^m examples such that the points of each invariant under every isomorphism can be identified.

J. W. Tukey.

Sierpiński, Wacław. Sur un théorème de M. Tarski concernant les alephs. Fund. Math. 34, 6–8 (1947).

It was announced without proof by A. Tarski [Lindenbaum and Tarski, C. R. Soc. Sci. Lett. Varsovie. Cl. III. 19, 299–300 (1926)] that the equivalence of the following statements can be established without the use of the axiom of choice: (a) $\mathfrak{m}$ is an aleph, (d) $[\mathfrak{m}+\aleph(\mathfrak{m})]-\mathfrak{m}=\aleph(\mathfrak{m})$. Here $\aleph(\mathfrak{m})$ is the smallest aleph $\mathfrak{n}$ for which the inequality $\mathfrak{n}\leqq\mathfrak{m}$ does not hold.

The author shows that if Tarski's statement is true, then it can also be proved without using the axiom of choice that the inequality $\aleph_0\leqq\mathfrak{m}$ holds for every infinite cardinal $\mathfrak{m}$. However, since it has been shown by R. Doss [J. Symbolic Logic 10, 13–15 (1945); these Rev. 7, 46] that this inequality cannot be proved without using the axiom of choice, we may conclude that Tarski's theorem is false.

B. Jónsson (Providence, R. I.).

Sierpiński, W. Sur un théorème de A. Lindenbaum. Ann. of Math. (2) **48**, 641–642 (1947).

Let m be an infinite cardinal, Ω_m the initial number belonging to m. Two sets A and B are called m-equivalent if

$$A=\bigcup_{\alpha<\Omega_m} A_\alpha,\quad B=\bigcup_{\alpha<\Omega_m} B_\alpha;\quad A_{\alpha_1}\cap A_{\alpha_2}=0,\quad B_{\alpha_1}\cap B_{\alpha_2}=0;$$

and A_α is congruent to B_α for all $\alpha<m$. The author constructs, without the axiom of choice, 2^c subsets of the line, so that for any $m<c$ no two of them are m-equivalent. This strengthens results of Lindenbaum and the reviewer.

P. Erdös (Syracuse, N. Y.).

Sierpiński, Wacław. Sur un ensemble plan qui se décompose en $2^{\aleph_0}$ ensembles disjoints superposables avec lui. Fund. Math. **34**, 9–13 (1947).

Proof, without use of the axiom of choice (multiplicative axiom) of the following theorem. If E is any given set of real numbers in the interval $0\leqq t<1$, and $0\in E$, one may effectively define a planar set P, and a decomposition τ of P into k disjoint subsets, where k is the cardinal of E, such that every constituent of τ is congruent to P (in the Euclidean sense of superposition). A. Lindenbaum announced this result without proof [Lindenbaum and Tarski, C. R. Soc. Sci. Lett. Varsovie. Cl. III. **19**, 299–330 (1926), p. 327] as a mere existence theorem, asserting its provability with the aid of the axiom of choice. The proof utilizes the function

$$f(t)=\sum_{n=0}^{\infty} 2^{2^{Ent}}-2^{n^2}$$

where Ex denotes the largest integer not exceeding x, and a result of von Neumann on the algebraic independence of any finite set of distinct values of $f(t)$, for $t>0$. By a modification of the argument, the author shows how to define effectively a nondenumerable planar set which is the disjoint sum of 2 subsets, each congruent with it, thus solving a problem of Steinhaus.

H. Blumberg.

Sierpinski, Waclaw. Sur les types d'ordre de puissance du continu. Actas Acad. Ci. Lima **9**, 267–269 (1946).

Effective definition of a family of 2^c sets of real numbers such that any pair of these sets, when ordered, as usual, according to the magnitude of their elements, are of different order type.

H. Blumberg (Columbus, Ohio).

Sierpinski, Waclaw. Sur l'existence d'un ensemble indénombrable à propriété λ'. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 353–354 (1940).

On dit qu'un ensemble linéaire E jouit de la propriété λ' s'il existe pour tout dénombrable D un H qui est un G_δ tel que $E\cap H\subset D\subset H$. Utilisant l'hypothèse du continu, l'auteur a démontré précédemment l'existence d'ensembles non dénombrables ayant la propriété λ' [C. R. Soc. Sci. Lett. Varsovie. Cl. III. **30**, 257–259 (1937)]. Il remarque maintenant que cette existence est une conséquence immédiate du théorème de Hausdorff suivant dont la démonstration n'utilise que l'axiome du choix [Fund. Math. **26**, 241–255 (1936), p. 248]: la droite est la réunion d'une suite transfinie croissante de type Ω d'ensembles G_δ.

R. de Possel (Alger).

Sierpiński, Wacław. Deux théorèmes sur les familles de transformations. Fund. Math. **34**, 30–33 (1947).

The author proves the following theorem: if M is an infinite set of cardinal number m and F is a family having cardinal number m of functions defined on M (with values in M or not in M), then there exists a family Φ consisting of 2^m subsets of M such that $f(E)\neq H$ for all $E, H\in\Phi$ such that $E\neq H$ and all $f\in F$. The proof does not consider one special case. A second theorem of the same general character is proved.

E. Hewitt (Chicago, Ill.).

Hsu, L. C. Abstract theory of inversion of iterated summations. Duke Math. J. **14**, 465–473 (1947).

A la suite de P. Hall [Quart. J. Math., Oxford Ser. **2**, 134–151 (1936)] et de L. Weissner [Trans. Amer. Math. Soc. **38**, 485–492 (1935)], l'auteur généralise la fonction de Möbius pour une famille de sous-ensembles ordonnées par inclusion. Soit $\tau^{(m)}(A, B)$ le nombre de chaînes de longueur m (avec répétition) allant de A à B. Soit de même $\lambda^{(m)}(A, B)$ le nombre de chaînes sans répétition. La fonction $\mu_m(X, K)$ est définie par

$$\sum_{H\subseteq X\subseteq K}\tau^{(m)}(H, K)\mu_m(X, K)=\begin{cases}1, & H=K,\\ 0, & H\neq K.\end{cases}$$

On a $\mu_m(X, K)=\sum_s(-1)^s\binom{m+s-1}{s}\lambda^{(s)}(X, K)$. Cette fonction permet de trouver des formules d'inversion pour les relations

$$\rho_{k+1}(H_{k+1})=\sum_{H_k\subseteq H_{k+1}}\rho_k(H_k),$$

$$\rho_{k+1}(H_{k+1})=\sum_{H_k}\phi_k(H_k, H_{k+1})\rho_k(H_k),$$

ϕ_k donné, $\phi_k(H, H)\neq 0$. On obtient des propriétés duales en remplaçant $\subseteq$ par $\supseteq$. Dans le cas où l'ensemble de base est produit direct de deux ensembles partiels et chaque sous ensemble produit direct de ses projections, la fonction μ_m est le produit des deux fonctions partielles. Applications, en particulier aux fonctions de probabilité.

J. Kuntzmann (Grenoble).

Farah, Edison. Zorn's theorem. Bol. Soc. Mat. São Paulo **1**, 19–34 (1946). (Portuguese)

Proof that the axiom of choice implies Zorn's principle; exposition of related ideas.

J. W. Tukey.

Orlicz, W. Une généralisation d'un théorème de MM. S. Banach et S. Mazur. Ann. Soc. Polon. Math. **19** (1946), 62–65 (1947).

Let $V_p(f)$ denote, for a function $f(x)$ defined in $a\leqq x\leqq b$, the least upper bound of the sums $\sum|f(x_i)-f(x_{i-1})|^p$ for subdivisions $a=x_0<x_1<\cdots<x_k=b$; write $V(f)$ for $V_1(f)$. Moreover, given a sequence of functions $f_n(x)$ in $a\leqq x\leqq b$, let $S_{n1}(x)=f_1(x)\epsilon_1(t)+\cdots+f_n(x)\epsilon_n(t)$, where $\epsilon_1(t)$, $\epsilon_2(t)$, $\cdots$ is the sequence of Rademacher functions in $0\leqq t\leqq 1$. The author proves the following theorem (I). Suppose that, for all n and almost every t, $V_p(S_{n1})\leqq M$; then there is a constant $C_p\leqq 8^p$ such that, for every subdivision $a=x_0<x_1<\cdots<x_k=b$, (a) if $p\geqq 2$,

$$\sum_n\Big(\sum_i|f_n(x_i)-f_n(x_{i-1})|^p\Big)\leqq MC_p,$$

(b) if $1\leqq p\leqq 2$,

$$\sum_n\Big(\sum_i|f_n(x_i)-f_n(x_{i-1})|^p\Big)^{2/p}\leqq MC_p.$$

[The proof of (b) contains a misprint (twice): the exponents $(2-p)/2$ and $2/(2-p)$ are interchanged.] As a corollary of (I), the author obtains (II). If, for all n and almost every t, $V(S_{n1})\leqq M$, then $\sum V^2(f_n)\leqq 8M$. This may be regarded as a greatly strengthened form of a theorem of Banach and Mazur [Studia Math. **4**, 100–112 (1933), p. 108]; in the latter, the hypothesis is equivalent to the one obtained on substituting "every" for "almost every," and the conclusion is weakened to $V(f_n)\to 0$.

L. C. Young.

Orlicz, Wladyslaw. Sur les fonctions continues non dérivables. Fund. Math. **34**, 45–60 (1947).

The results obtained are intermediate between the actual construction of continuous nondifferentiable functions (like Weierstrass's function) and pure existence theorems (e.g., theorems of the type that in a suitably chosen complete metric space of continuous functions all functions with the exception of a set of first category are nondifferentiable). A typical example is the following theorem. If $\varphi(x)$ is a continuous periodic function (not a constant) with continuous derivative for all x, $\alpha_n>0$, $\sum\alpha_n$ convergent, $0<\beta_1<\cdots<\beta_n<\cdots$, $\beta_n\to\infty$, $\alpha_n\beta_n>c>0$, and $\eta_n=0, 1$, then the function $\phi_\eta(x)=\sum_{n=1}^{\infty}\eta_n\alpha_n\varphi(\beta_n x)$ does not possess a derivative on the right at any x for all sequences $\eta=\{\eta_n\}$ of a residual set in the space of all sequences η (where distance is defined by $d(\eta', \eta'')=\sum_{n=1}^{\infty}2^{-n}|\eta_n'-\eta_n''|$). Other types of functions investigated are $\phi_\epsilon(x)=\sum_{n=1}^{\infty}\epsilon_n\alpha_n\varphi(\beta_n x)$, where $\epsilon_n=\pm1$, and $f_\epsilon(x)=\sum_{n=1}^{\infty}\epsilon_n f_n(x)$, $f_\eta(x)=\sum_{n=1}^{\infty}\eta_n f_n(x)$, where $f_n(x)$ are continuous and $\sum_{n=1}^{\infty}|f_n(x)|$ is uniformly convergent for $a\leqq x\leqq b$. *F. A. Behrend* (Melbourne).

Mirguet, Jean. Les paratingents supérieurs des orthosurfaces. Revue Sci. **85**, 67–72 (1947).

Soit M un point d'accumulation pour un ensemble E, $S_r(M)$ une sphère de centre M et de rayon r. Les positions limites, quand $r\to0$, des droites supportant n points ($n>2$) de $E-M$, situés dans $S_r(M)$, sont les paratingents supérieurs de E en M. Un ensemble fermé, tel qu'une droite a échappe en M au paratingent de E, est orthosurface en M, si toute droite a' tendant vers a en lui restant parallèle, contient un point de E tendant vers M (au moins quand la distance $\overline{aa'}$ devient suffisamment petite). Un ensemble qui est orthosurface en chacun de ses points est une orthosurface. L'auteur étudie au § 1 les surfaces dont le paratingent général n'a pas d'intérieur, au § 2 celles dont le paratingent supérieur n'a pas d'intérieur: c'est-à-dire les surfaces dont aucun paratingent (respectivement paratingent supérieur) est limite d'autres paratingents (respectivement paratingents supérieurs). Au § 3 l'auteur étudie enfin les orthosurfaces à paratingent supérieur fini, c'est-à-dire qu'en tout point le paratingent supérieur est constitué d'un nombre fini de droites. *T. Viola* (Rome).

Besicovitch, A. S. Corrigenda to the paper "A general form of the covering principle and relative differentiation of additive functions. II." Proc. Cambridge Philos. Soc. **43**, 590 (1947).

The paper appeared in the same Proc. **42**, 1–10 (1946); these Rev. **7**, 281.

Kryloff, V. I. Sur l'existence des dérivées généralisées des fonctions sommables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 375–378 (1947).

L'auteur définit une notion de dérivée généralisée. Soit ∂ un symbole de dérivation, ∂^* son adjoint ou transposé (rappelons que l'adjoint de $\partial^{k_1+\cdots+k_n}/\partial x_1^{k_1}\cdots\partial x_n^{k_n}$ est $(-1)^{k_1+\cdots+k_n}\partial^{k_1+\cdots+k_n}/\partial x_1^{k_1}\cdots\partial x_n^{k_n}$); la fonction $u(x_1, \cdots, x_n)$ a pour dérivée généralisée la fonction $v(x_1, \cdots, x_n)$ si l'on a

$$\int\cdots\int vw dx_1\cdots dx_n=\int\cdots\int u\partial^* w dx_1\cdots dx_n,$$

quelle que soit la fonction $w(x_1, \cdots, x_n)$, nulle en dehors d'un compact et à dérivées d'ordre assez élevé continues (u et v sont sommables sur tout compact, et définies seulement presque partout). Le théorème démontré est le suivant: si u a un laplacien itéré $\Delta^k u$, de p-ième puissance sommable sur tout compact, elle a des dérivées de tout ordre inférieur à $2k$, et toute dérivée d'ordre s de u est de puissance q-ième sommable sur tout compact, pour tout q vérifiant $(1/q)\geqq(1/p)-((2k-s)/n)$ si $p>1$, le signe $\geqq$ devant être remplacé par $>$ si $p=1$. Le théorème est donné sous une forme un peu plus générale, u étant éventuellement remplacé par un système de n fonctions $u_1, \cdots, u_n$, et le laplacien par d'autres symboles analogues. La démonstration, donnée dans ses grandes lignes, utilise la régularisation par moyennes, la solution fondamentale des équations aux dérivées partielles utilisées, et une inégalité de Soboleff [Rec. Math. [Mat. Sbornik] N.S. **4(46)**, 471–497 (1938)]. [La définition de la dérivée généralisée est aussi celle utilisée par L. Schwartz dans la théorie des distributions [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); ces Rev. **8**, 264] dans laquelle d'ailleurs on peut donner une généralisation des théorèmes indiqués.] Des théorèmes plus particuliers [cités par l'auteur] avaient été donnés par Soboleff, Kondrachov et Friedrichs.

L. Schwartz (Nancy).

Faedo, Sandro. Alcuni nuovi criteri di eguale continuità per le funzioni di più variabili. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 188–202 (1947).

The author gives three sets of sufficient conditions (differing from those previously given by Tonelli) for the equicontinuity of sets of functions $f(x, y)$ of two variables, which are known to be absolutely continuous in the sense of Tonelli. The functions of accumulation of such sets are shown to be absolutely continuous in the same sense. The author concludes by indicating very briefly some extensions of the criteria to functions of n variables. *L. M. Graves.*

Schaerf, H. M. On the continuity of measurable functions in neighborhood spaces. Portugaliae Math. **6**, 33–44; errata, 66 (1947).

Let R be a neighborhood space in which a measure m is defined. A function f with values in a topological space T is called measurable if $f^{-1}(G)$ is measurable whenever G is open. The measure m is said to have property (a) if the measurable closed sets in R are dense in the space of measurable sets, with the metric $\rho(A, B)=m(A\Delta B)$. It is shown that, if m has property (a), T has second countability, $f(x)$ is measurable on $E\subset R$, and $\epsilon>0$, then $f(x)$ is continuous on some set $E_\epsilon\subset E$ with $m(E-E_\epsilon)<\epsilon$. The set E_ϵ can be taken closed if and only if every measurable set can be internally approximated by a closed measurable set. If every closed set in R is G_δ then every finite Borel measure in R has property (a). When R and T are uniform spaces, a sufficient condition for $f(x)$ to be uniformly continuous on E_ϵ is obtained and applied to solutions of the functional equation $f(xy)=f(x)f(y)$ in topological groups [but cf. the errata]. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Ottaviani, G. Sulla indipendenza delle funzioni misurabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 393–398 (1947).

General remarks on pairs of Lebesgue measurable functions $f(t)$, $g(t)$ defined in the interval (0, 1) which are independent, that is, which satisfy the equation $m\{f\epsilon A, g\epsilon B\}=m\{f\epsilon A\}\cdot m\{g\epsilon B\}$ for all Borel sets A, B. If g is absolutely continuous and nonconstant, it is shown

that f coincides almost everywhere with a function which takes on only finitely many values. *J. L. Doob.*

Doubrovsky, V. On some properties of completely additive set functions and their application to generalization of a theorem of Lebesgue. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 317–329 (1947). (Russian. English summary)

The paper deals with sequences of completely additive real set-functions in an abstract space A. We denote by S the class of such set-functions $\Phi(E)$ defined for an appropriate fixed system of subsets E of A; and by S_M the subclass of S for which $\Phi(E)$ is absolutely continuous with respect to a measure function $M(E)$. The author defines, for sequences $\{\Phi_n\}$ of such set-functions, the notions of uniform additivity and of uniform (or "equi-gradual") absolute continuity, the uniformity being in respect of the parameter n. The corresponding classes of these sequences $\{\Phi_n\}$ may be denoted by $\{S\}$ and by $\{S_M\}$. As the author observes, it is easily seen that $\{S\}$ includes $\{S_M\}$ and that if $\{\Phi_n\}$ belongs to either of these classes so does the sequence $\{\bar{\Phi}_n\}$ of the total variations $\bar{\Phi}_n$ of Φ_n. Similar remarks hold when n is replaced by a parameter α.

The main results of the paper are as follows. (A) If $\Phi_n \varepsilon S$, weak convergence of the sequence $\Phi_n(E)$ implies $\{\Phi_n\} \varepsilon \{S\}$; (B) if $\Phi_n \varepsilon S_M$ the relation $\{\Phi_n\} \varepsilon \{S\}$ implies $\{\Phi_n\} \varepsilon \{S_M\}$. Their combination includes a generalization of the following classical theorem due to Lebesgue: suppose that the functions $f_n(x)$ summable in $0 \leqq x \leqq 1$ are such that for each measurable subset E of $(0, 1)$ the integrals $\Phi_n(E) = \int_E f_n(x)dx \to 0$ as $n \to \infty$; then we have, uniformly in n, $\Phi_n(E) \to 0$ as $M(E) \to 0$, where $M(E)$ denotes the Lebesgue measure of E. The paper concludes with an application to Euclidean n-dimensional space. *L. C. Young.*

Izumi, Shin-ichi, Matuyama, Noboru, and Orihara, Masae. An abstract integral. V. Proc. Imp. Acad. Tokyo **18**, 45–49 (1942). [MF 14740]

Nakamura, Masahiro. An abstract integral. VI. Proc. Imp. Acad. Tokyo **18**, 50–52 (1942). [MF 14741]

Izumi, Shin-ichi. An abstract integral. VII. Proc. Imp. Acad. Tokyo **18**, 53–56 (1942). [MF 14742]

Orihara, Masae, and Sunouchi, Gen-ichirô. An abstract integral. VIII. Proc. Imp. Acad. Tokyo **18**, 535–538 (1942). [MF 14776]

Matsuyama, Noboru. An abstract integral. IX. Proc. Imp. Acad. Tokyo **18**, 539–542 (1942). [MF 14777]

Izumi, Shin-ichi. An abstract integral. X. Proc. Imp. Acad. Tokyo **18**, 543–547 (1942). [MF 14778]

[For paper IV, by Izumi, see the same Proc. **17**, 1–4 (1941); these Rev. **2**, 355.] The purpose of this chain of papers is to define certain linear operations (called integrals) with various extendability, uniqueness and invariance properties. The following list describes the domain of the principal operation studied in paper number N and indicates in each case the name of the mathematician to whom the authors attribute the origin of the theory being generalized. V. [Banach] A partially ordered linear space. VI. [Freudenthal] Functions from a lattice to a metric commutative semigroup. VII. [Bochner] Functions from a Euclidean space to a sigma-complete vector lattice. VIII. [Daniell] Functions from an abstract set to a complete regular vector lattice. IX. [Goldstine] A sigma-complete vector lattice. X. [Denjoy] Real functions of a real variable for which the inverse of an axiomatically defined derivative exists. *P. R. Halmos* (Princeton, N. J.).

James, R. D., and Gage, Walter H. A generalized integral. Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 25–35 (1946).

In certain problems (for example, in the theory of uniqueness of trigonometric series) it is desirable to define a second integral of a given function $g(x)$ without defining the first integral. In this paper the authors develop the theory of such a second integral which they call a second Perron integral. Given a function $g(x)$, let

$$\Delta_h^2 g(x) = g(x+h) + g(x-h) - 2g(x),$$

and let $\bar{D}^2 g$ and $\underline{D}^2 g$ denote, respectively, the limits superior and inferior of $\Delta_h^2 g(x)/h^2$ for $h \to 0$. If $\bar{D}^2 g = \underline{D}^2 g$, their common value will be denoted by $D^2 g$ (the second generalized derivative). Given a function $f(x)$ in an interval (a, c), a continuous function $M(x)$ is called a major function (of the second order) for $f(x)$ if (i) $M(a) = M(c) = 0$, (ii) $\underline{D}^2 M \geqq f(x)$, (iii) $\underline{D}^2 M > -\infty$ (condition (ii) may be omitted at a denumerable set of points if we assume that $M(x)$ is smooth, that is, if $\Delta_h^2 M(x) = o(h)$). Similarly we define a minor function $m(x)$. A function $f(x)$ defined in an interval (a, c) is said to be integrable (a, b, c), where $a < b < c$, if for every $\epsilon > 0$ there are a major function $M(x)$ and a minor function $m(x)$ such that $0 \leqq m(b) - M(b) < \epsilon$.

It is shown that if f is integrable (a, b, c) then there is a function $F(x)$ defined in (a, c) which is the least upper bound of all $M(x)$ and at the same time the greatest lower bound of all $m(x)$ (for a given f) in (a, c). Of all the properties of F we mention the following ones: (a) if f is integrable (a, b, c) then it is integrable (a, x, c) for every x in (a, c). (b) If f is integrable (a, b, c), then at almost every point x in (a, c) the derivative D^2F exists and equals f. (c) If a continuous function $G(x)$ has a finite D^2G at every point interior to (a, c), then D^2G is integrable (a, b, c). [The use of major and minor functions for the purposes of second integration is implicit in many previous papers. See de la Vallée Poussin, Acad. Roy. Belgique. Bull. Cl. Sci. 1912, 702–718. It is shown there that (using the terminology of the present paper) every Lebesgue integrable function is also integrable (a, b, c). See also Burkill, Proc. London Math. Soc. (2) **34**, 314–322 (1932); Marcinkiewicz and Zygmund, Fund. Math. **26**, 1–43 (1936); and the book of Denjoy, Leçons sur le Calcul des Coefficients d'une Série Trigonométrique, Gauthier-Villars, Paris, 1941; these Rev. **8**, 260.] *A. Zygmund* (Chicago, Ill.).

Ridder, J. Ueber Definitionen von Perron-Integralen. II. Nederl. Akad. Wetensch., Proc. **50**, 474–483 = Indagationes Math. **9**, 280–289 (1947).

[For part I cf. the same Proc. **50**, 369–377 = Indagationes Math. **9**, 227–235 (1947); these Rev. **8**, 506.] For a function $f(x)$ major and minor functions ψ_r, φ_r are defined which satisfy $-\infty < D_+\psi_r \geqq f(x)$, $\infty > D^+\varphi_r \leqq f(x)$. These functions lead in the usual way to a definition of integration which is shown to be equivalent to the Perron-Bauer definition. A similar statement holds for major and minor functions ψ_l, φ_l, where left hand derivation only is considered. The author then defines a quadruple of generalized major and minor functions Ψ_r, Φ_r, Ψ_l, Φ_l, where $[a, b]$ is covered by a denumerable set E and a denumerable set of perfect sets $\{P_j\}$ and at each point of P_j

$$-\infty < D_{-P_j}\Psi_l \geqq f, \quad -\infty < D_{+P_j}\Psi_r \geqq f,$$
$$\infty > D^-_{P_j}\Phi \leqq f, \quad \infty > D^+_{P_j}\Phi \leqq f,$$

differentiation being over the set P_j. These functions lead in a way similar to that used by McShane [Integration, Princeton University Press, 1944; these Rev. **6**, 43] to a

definition of integration, $\mathfrak{P}$-integration, which the author shows to be equivalent to the generalized Perron-Bauer definition. It is then shown that this definition is equivalent to one obtained by using only generalized right major functions Ψ_r, and generalized left minor functions Φ_r, or by using only generalized left major functions and generalized right minor functions. The paper concludes with a determination of the formulas for integration by substitution and for integration by parts for $\mathfrak{P}$-integrals.

R. L. Jeffery (Kingston, Ont.).

Ridder, J. Eine Bemerkung über das Mass in Strukturen. Nederl. Akad. Wetensch., Proc. **50**, 607–611 = Indagationes Math. **9**, 315–319 (1947).

A real, possibly infinite, monotone function on a Boolean algebra is a "mass" if it is not identically infinite or zero and if the null element of the algebra is measurable (in the sense of Carathéodory). If the algebra is a closure algebra (in the sense of McKinsey and Tarski [Ann. of Math. (2) **45**, 141–191 (1944); these Rev. **5**, 211]) then a necessary and sufficient condition that every closed set is measurable is that the mass is additive for every pair of sets whose union has finite mass and of which one is disjoint from the closure of the other. An analogue of this result for countably additive algebras is also valid. *P. R. Halmos.*

Loomis, L. H. On the representation of σ-complete Boolean algebras. Bull. Amer. Math. Soc. **53**, 757–760 (1947).

Every σ-complete Boolean algebra is σ-isomorphic to a σ-complete Boolean algebra of point sets modulo a σ-ideal in that algebra. *P. R. Halmos* (Princeton, N. J.).

Pauc, Christian. Compléments à la représentation ensembliste d'une algèbre et d'une σ-algèbre booléennes. C. R. Acad. Sci. Paris **225**, 219–221 (1947).

Let B be a Boolean algebra and S the associated totally-disconnected compact Boolean space (whose neighborhoods are open and compact, and form a Boolean algebra isomorphic to B). If μ is a finitely additive measure on B then the transplanted measure $\bar{\mu}$, finitely additive on the neighborhoods of S, has a completely additive extension to the Borel field F generated by these neighborhoods. The author announces the following theorem. Let A be a σ-complete Boolean algebra and μ a completely additive measure on A which is effective (different from zero on every nonzero element of A). If B is any dense Boolean subalgebra of A, then A is σ-isomorphic to the Borel field F in the Boolean space of B, reduced modulo the ideal of null sets of the measure $\bar{\mu}$ on F. *L. H. Loomis* (Cambridge, Mass.).

***Kappos, D. A. Ortsfunktionen von zwei Veränderlichen und Doppelintegrale in Booleschen Algebren.** Ber. Math.-Tagung Tübingen 1946, pp. 87–89 (1947).

A sketchy statement, without proof, of an analogue of Fubini's theorem for Boolean algebras. *P. R. Halmos.*

Theory of Functions of Complex Variables

***Macrobert, Thomas M. Functions of a Complex Variable.** 3d ed. Macmillan and Co., Limited, London, 1947. xv+390 pp. $4.50.

The first edition of this well-known text appeared in 1917, the second in 1933. Its special features are an emphasis on geometrical methods, extensive discussion of special functions and second-order differential equations, and a profusion of illustrative examples. The new edition differs from the second by the addition of an appendix [20 pages] on generalized hypergeometric functions and a collection of 125 new miscellaneous examples. *R. P. Boas, Jr.*

Kasner, Edward, and De Cicco, John. Partial differential equations related to rational functions of a complex variable. Duke Math. J. **14**, 339–348 (1947).

It is shown that the polynomials $\varphi(x, y)$ and $\psi(x, y)$ are the numerators of the real and imaginary parts of a rational function of the complex variable $z=x+iy$ if and only if the function $\log(\varphi+i\psi)$ is harmonic. It is shown further that a polynomial $\varphi(x, y)$ is the numerator of the real (or imaginary) part of a rational function of z if and only if the function $\log[\varphi^2\Delta\log\varphi]$ is harmonic. *E. F. Beckenbach.*

***Morse, Marston. Topological Methods in the Theory of Functions of a Complex Variable.** Annals of Mathematics Studies, no. 15. Princeton University Press, Princeton, N. J., 1947. iv+145 pp. $2.50.

L'auteur expose sous une forme nouvelle une partie des résultats qu'il a obtenus personnellement ou en collaboration avec M. Heins dans l'étude des transformations générales qui possèdent les propriétés topologiques des fonctions analytiques. Ce sont les transformations intérieures de Stoïlow [Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques, Gauthier-Villars, Paris, 1938], considérées aussi par Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., t. 28, New York, 1942; ces Rev. **4**, 86].

Dans le chapitre I, l'auteur introduit les fonctions pseudo-harmoniques déduites des fonctions harmoniques comme les transformations intérieures le sont des fonctions méromorphes. Il étudie et classe les singularités d'une telle fonction U dans son domaine de définition G et sur sa frontière B. L'étude de la caractéristique d'Euler du domaine E_c dans lequel $U<c$ conduit à la formule $2-v=m-S-s$ valable lorsque U n'a pas de singularité logarithmique polaire; v est le nombre des courbes constituant B, m le nombre des points où U a un minimum relatif, S et s les nombres de points critiques de U convenablement définis (saddle points) dans G et sur B. Si la frontière de G est formée de m courbes de Jordan, la formule vaut encore s'il y a M poles logarithmiques dans G, à condition d'ajouter M au second membre. On peut alors écrire $M-S=2-v+I$; I est l'indice de la frontière, égal ici à $s-m$. Le cas où U est constant sur certaines parties de B est alors examiné.

Au chapitre II, l'indice I est calculé de diverses façons moyennant des hypothèses sur la frontière B et sur les valeurs de U sur B (frontière de Jordan avec un nombre fini d'extrema de U, conditions de dérivabilité pour les antécédents de B et U). Les démonstrations utilisent les notions d'ordre de distance de deux courbes dues à Fréchet.

Au chapitre III sont introduites les notions de courbes localement simples et d'ordre angulaire de telles courbes. Pour une transformation intérieure $w=f(z)$ qui est biunivoque au voisinage de tout point de B et qui ne s'annule pas sur B, les théorèmes du chapitre II conduisent à la formule fondamentale $n(0)+n(\infty)-\mu=2-v+q(0)-p$; $n(a)$ est le nombre des points de G où $f(z)=a$, μ le nombre des points de ramification de l'inverse de $f(z)$, $q(c)$ la somme des ordres par rapport à c et p la somme des ordres angulaires des courbes transformées de B. De la formule fondamentale

découle l'égalité $n(a)-n(b)=q(a)-q(b)$, dont le champ d'application est élargi, et aussi $2n(a)=\mu+2-v+2q(a)-p$.

Après avoir montré au début du chapitre IV la nécessité d'hypothèses précises sur la frontière B, l'auteur étend notamment cette dernière formule au cas où la transformée de B est supposée seulement localement simple, la signification de μ étant convenablement précisée; c'est là un résultat nouveau.

Au chapitre V, l'auteur étudie en détail certaines classes de déformation des courbes localement simples; la paramétrisation des courbes est faite au moyen d'une extension de la notion de longueur définie antérieurement par Whitney [Ann. of Math. (2) **34**, 244–270 (1933)]. Les déformations continues admises sont étudiées; il est montré notamment que: (a) une courbe localement simple fermée peut être déformée en une courbe arbitrairement voisine formée d'arcs analytiques; (b) deux courbes localement simples orientées fermées ayant le même ordre angulaire peuvent être déformées l'une en l'autre. Ensuite est introduit le produit de deux courbes localement simples; enfin, en vue de l'application à l'étude des classes de déformation des transformations intérieures sont étudiées les déformations de courbes au cours desquelles aucune courbe ne passe par l'origine. Des indications sur les nouveaux résultats du mémoire écrit en collaboration avec M. Heins [Acta Math. **79**, 51–103 (1947); ces Rev. **8**, 507] et sur les problèmes qu'ils suggérent terminent ce volume. *G. Valiron.*

Roussel, André. Une généralisation du développement de Taylor. C. R. Acad. Sci. Paris **225**, 23–24 (1947).

Let $g(x, h)$ be a holomorphic function of the complex variables x, h in the circles $|x-a|\leqq R$, $|h|\leqq R'$, with $g(x,0)=0$. Define the sequence $\{g_n(x,h)\}$ $(n=0,\pm1,\pm2,\cdots)$ as follows: $g_0(x,h)=g(x,h)$; for $n>0$,

$$g_n(x,h)=\int_0^h\left\{\frac{\partial}{\partial x}g_{n-1}(x,h)\right\}dh,$$

$$g_{-n}(x,h)=\int_a^x\left[\frac{\partial}{\partial h}g_{-(n-1)}(x,h)\right]_{h=0}^{h=h}dx.$$

It is shown that the series $\sum_{-\infty}^{\infty}g_n(x,h)$ converges in the region $|x-a|+|h|<\min\{R,R'\}$ to the function $f(x+h)-f(x)$, where $f(x)=g(a,x-a)+\int_0^{x-a}g_x'(x-t,t)dt$. The choice $g(x,h)=h\varphi'(x)$ leads to Taylor's series for $\varphi(x+h)-\varphi(x)$. *I. M. Sheffer* (State College, Pa.).

Roussel, André. Sur certaines généralisations des séries de Taylor. C. R. Acad. Sci. Paris **225**, 348–349 (1947).

Let $g(x,y,h,k)$ be a function of the complex variables x, y, h, k holomorphic for $|x-a|<R_1$, $|y-b|<R_2$, $|h|<R_3$, $|k|<R_4$. Let $g(x,y,0,0)\equiv0$. Let $\phi_{m,n}(x,y,h,k)$ be obtained from $g(x,y,h,k)$ by the following process. (1) If m is a positive integer, let g be integrated m times with respect to h, from 0 to h, and then differentiated m times with respect to x. If m is a negative integer, let g be integrated $(-m)$ times with respect to x from a to x, and then differentiated $(-m)$ times with respect to h. (2) If n is a positive integer, integrate n times with respect to k, from 0 to k, the result obtained in (1), and then differentiate n times with respect to y. If n is a negative integer, integrate $(-n)$ times with respect to y, from b to y, the result obtained in (1), and then differentiate $(-n)$ times with respect to k. Let $g_{0,0}(x,y,h,k)=g(x,y,h,k)$; $g_{m,n}(x,y,h,k)=\phi_{m,n}(x,y,h,k)-\phi_{m,n}(x,y,0,0)$. Then for $|x-a|+|h|<\rho$, $|y-b|+|k|<\rho$, $0<\rho<\min\{R_1,R_2,R_3,R_4\}$, the series

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}g_{m,n}^{(x,y,h,k)}-\sum_{m=-\infty}^{\infty}g_{m,0}^{(x,y,h,0)}-\sum_{n=-\infty}^{\infty}g_{0,n}^{(x,y,0,k)}$$

is absolutely and uniformly convergent to $f(x+h,y+k)-f(x,y)$, where

$$f(x,y)=g(a,b,x-a,y-b)+\int_0^{x-a}g_x'(x-t,b,t,y-b)dt+\int_0^{y-b}g_y'(a,y-t,x-a,t)dt+\int_0^{x-a}dt\int_0^{y-b}g_{xy}''(x-t,y-\tau,t,\tau)d\tau.$$

Taylor's series for $F(x+h,y+k)-F(x,y)$ is a special case where $g(x,y,h,k)=F_x'(x,y)h+F_y'(x,y)k$. The theorem may be extended to any number of variables.

M. S. Robertson (New Brunswick, N. J.).

Seleznev, A. Une généralisation d'un théorème d'Hadamard sur les séries de Taylor admettant le cercle de convergence comme coupure. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 311–315 (1947). (Russian. French summary)

The author calls "generalized star" E_R of $f(z)$ the set of points of Mittag–Leffler's star of $f(z)$ which are contained in the circle $|z|<R$. He proves the following theorem. If $f(z)=\sum a_nz^n$ $(0<r^{-1}=\limsup|a_n|^{1/n}<\infty)$, it is possible to change the moduli of an infinity of coefficients so that the generalized star E_R $(R\geqq r)$ is a natural boundary for the new function. *S. Mandelbrojt* (Houston, Tex.).

Ilieff, Ljubomir. Analitisch nichtfortsetzbare Potenzreihen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 67–81 (1946). (Bulgarian. German summary)

The author's principal theorems are as follows. (A) Let $f(z)=\sum_0^\infty c_nz^n$ with the unit circle as circle of convergence. Let the c_n be bounded and have only a finite number of limit points; the c_n can be separated into subsequences which converge to the various limit points. Then either these subsequences are ultimately distributed periodically or the unit circle is a natural boundary for $f(z)$ [however, it may be a natural boundary even if the subsequences are distributed periodically]. This generalizes a theorem of Szegö [see, e.g., Dienes, The Taylor Series, Oxford University Press, 1931, p. 324] in which there are only a finite number of different coefficients. The author's proof depends on an earlier theorem of his which was not available to the reviewer; however, the theorem can be proved [in a more general form in which the conclusion is that $f(z)$ is unbounded in every sector of the unit circle unless the coefficients are distributed periodically] by the method used by Duffin and Schaeffer to generalize Szegö's theorem [Amer. J. Math. **67**, 141–154 (1945); these Rev. **6**, 148].

(B) Let the c_n be bounded and let there be a sequence n_p such that $c_{n_p}\to\alpha\neq0$, while $c_{n_p+m}=0$, $m=1,2,\cdots,m_p$, $m_p\to\infty$. Then the unit circle is a natural boundary for $f(z)$. The proof depends on estimates used by Szegö in the proof of the theorem quoted above. [The theorem can also be proved by the method of Duffin and Schaeffer.]

R. P. Boas, Jr. (Providence, R. I.).

Cowling, V. F. On a class of Taylor series. Bull. Amer. Math. Soc. **53**, 544–547 (1947).

The author proves the following extension of a theorem of Mandelbrojt [Rice Institute Pamphlet **14**, 223–352

(1927)]. If $f(z)=\sum_0^\infty a_n z^n$ has rational coefficients, if there is an integer L such that $a_n L^n$ is integral for each n, and if $f(z)$ is regular in the closed half plane (including the point at infinity) $x \leqq L/2$, or outside of a certain open circle, then $f(z)$ is of the form $P(z)/(A-z)^{n_0}$, where $P(z)$ is a polynomial. The proof depends on the fact that a power series with integral coefficients cannot converge outside the unit circle without terminating. [These results also follow from theorems of Pólya and Carlson on the form of functions whose power series have integral coefficients and whose region of regularity has mapping radius greater than unity. See Bieberbach, Lehrbuch der Funktionentheorie, vol. 2, 2d ed., Teubner, Leipzig-Berlin, 1931, p. 327.]

R. C. Buck (Providence, R. I.).

Nassif, M. On the order of the product of simple sets of polynomials. Proc. Math. Phys. Soc. Egypt ~~3, 43–47 (1946).~~ 3 (1946), no. 2, 9–12 (1947)

Ghabbour, M. N. On convergence of the product of basic sets of polynomials. Amer. J. Math. 69, 583–591 (1947).

Let $p_n(z)$ and $q_n(z)$ be basic sets of polynomials [terminology as in J. M. Whittaker, "Interpolatory Function Theory," Cambridge University Press, 1935; "Series of Polynomials," Fouad I University, Cairo, 1943; these Rev. 8, 454]; the coefficients of z^k in p_n, q_n, respectively, are p_{nk}, q_{nk}. The product set of $\{p_n\}$ and $\{q_n\}$ (in this order) is the set whose coefficients are the elements of the product matrix $(p_{nk})\cdot(q_{nk})$.

The order of a basic set was defined by Whittaker ["I. F.T.," p. 11], who showed that a set of order ω represents all entire functions of order less than $1/\omega$. Nassif shows that, if $\{p_n\}$ and $\{q_n\}$ are simple sets (i.e., the nth polynomial is of degree n) and have respective orders ω_1, ω_2, their product has order not exceeding $1/(\omega_1+2\omega_2)$; this bound is best possible.

The set $\{p_n\}$ is called a Cannon set [Cannon, Proc. London Math. Soc. (2) 43, 348–365 (1937)] if $\lim N_n^{1/n}=1$, where N_n is the number of nonzero coefficients in the expansion of z^n in terms of $p_n(z)$. Let D_n be the degree of the polynomial of highest degree in the expansion of z^n. Ghabbour discusses the product of $\{p_n\}$ and $\{q_n\}$ when $\{p_n\}$ is a Cannon set and $\{q_n\}$ satisfies $\limsup D_n/n<\infty$. A variety of conditions, both necessary and sufficient, are given for the effectiveness of the product set in a given circle.

R. P. Boas, Jr. (Providence, R. I.).

Eweida, M. T. A note on the generalization of Taylor's expansion. Proc. Math. Phys. Soc. Egypt ~~3, 1–7 (1946).~~ 3 (1946), no. 2, 1–7 (1947)

The author applies the methods of J. M. Whittaker ["Interpolatory Function Theory," Cambridge University Press, 1935] to two expansions. The first is Abel's series,

$$F(z)=F(0)+\sum_{n=1}^{\infty}\frac{z(z-n\beta)^{n-1}}{n!}F^{(n)}(n\beta);$$

he corrects an erroneous statement of Watson [J. London Math. Soc. 3, 188–192 (1928)] and proves that the series represents $F(z)$ if $F(z)$ is an entire function of growth less than order one, type $\tau/|\beta|$, where τ is the positive root of $xe^{1+x}=1$. [This was proved by a different method by Gontcharoff, Rec. Math. [Mat. Sbornik] N.S. 1(42), 473–484 (1935); Détermination des fonctions entières par interpolation, Actual. Sci. Ind., no. 465, Hermann, Paris, 1937, p. 39; cf. also Schmidli, Über gewisse Interpolationsreihen, Dissertation, Eidgenössische Technische Hochschule in Zürich, 1942; these Rev. 4, 39.]

The second expansion is

$$F(z)=\sum_{k=0}^{\infty}G_n(z)F^{(n)}(a_n),$$

where $|a_n|\leqq k$ and $G_n(z)$ are the Gontcharoff polynomials. If $k\geqq\log 2$, the author shows that the series represents all entire functions of growth less than order 1, type $k/\log 2$. [The same result for any k follows from results of Ibragimoff, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 1939, 553–568; these Rev. 1, 310, and of Boas, Trans. Amer. Math. Soc. 48, 467–487 (1940); these Rev. 2, 80.]

R. P. Boas, Jr. (Providence, R. I.).

Slobodetzky, L. N. On the representation of regular functions by series of rational functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 56, 123–126 (1947).

The author considers series of the form

$$c_0+\sum_{n=1}^{\infty}c_n\prod_{k=1}^{n}\frac{z-a_k}{1-z\bar{a}_k},$$

where $|a_k|<1$, $\sum(1-|a_k|)=\infty$, and $\{a_k\}$ has only a finite number of limit points. Under various additional assumptions on the a_k, the behavior of the sum of the series and the possibility of representing a given analytic function are studied. No proofs are given. The case of one limit point was considered by W. and M. Gontcharoff [Uchenye Zapiski Moskov. Gos. Univ. Matematika 73, 3–22 (1944); these Rev. 7, 202].

R. P. Boas, Jr. (Providence, R. I.).

Ìbraghimoff, I., et Keldych, M. Sur l'interpolation des fonctions entières. Rec. Math. [Mat. Sbornik] N.S. 20(62), 283–291 (1947). (Russian. French summary)

Notation: $a_1, a_2, a_3, \cdots$ is a sequence of complex numbers, $|a_1|\leqq|a_2|\leqq\cdots$; $n(r)$ is the number of a_k with $|a_k|\leqq r$, $N(r)=\int_0^r u^{-1}n(u)du$; $f(z)$ is an integral function, $M(r)$ its maximum modulus on $|z|=r$; $H(z)$ is the Newton interpolation series of $f(z)$ with basic points a_k. (I) To any θ, $0<\theta<\frac{1}{2}$, there is a number $C(\theta)$ such that $H(z)$ converges uniformly to $f(z)$ if (1) $n(\theta r)>C(\theta)\log M(r)$ for $r>r(\theta)$. One may take $C(\theta)=\{\log(1-\theta^{-1})\}^{-1}+\epsilon$. (II) If $\theta>\frac{1}{2}$, there is an $f(z)$ satisfying (1) for every $C(\theta)>0$ and such that $H(z)$ does not converge to $f(z)$. Further, $\log M(r)<\exp(\log r)^{1+\epsilon}+K$. (III) If $0<\theta<1$ and

$$p_{nk}(z)=\prod_{k\neq j}^{n}(z-a_j)(a_k-a_j)^{-1}\prod_{j=1}^{n}\{1-\theta^2\bar{a}_k|a_n|^{-2}(a_j-z)\}$$

then $\sum_{k=1}^{n}f(a_k)p_{nk}(z)$ tends uniformly to $f(z)$, provided that $\lim N(\theta r)/\log M(r)>1$. This last condition is weaker than (1).

W. H. J. Fuchs (Liverpool).

Sunyer i Balaguer, Ferran. Sur la substitution d'une valeur exceptionnelle par une propriété lacunaire. C. R. Acad. Sci. Paris 224, 1609–1610 (1947).

Gap conditions (density and "maximal density") on the power series of the integral function $F(z)$ of integer order which ensure that the zeros of $F(z)-f(z)$ cannot be exceptional with respect to the proximate order of $F(z)$ for any meromorphic function $f(z)$ of lower order.

A. J. Macintyre (Aberdeen).

Sunyer i Balaguer, Ferran. Sur la substitution d'une valeur exceptionnelle par une propriété lacunaire. C. R. Acad. Sci. Paris 225, 21–23 (1947).

(1) The results of the paper reviewed above are extended to integral functions of infinite order.

(2) In the theorems of Schottky and Landau, the second of the conditions $f(z)\neq 0$, $f(z)\neq 1$ may be replaced by a gap hypothesis. Similarly for various tests for normality of function families. *A. J. Macintyre* (Aberdeen).

Walsh, J. L., and Nilson, E. N. Note on the degree of convergence of sequences of polynomials. Bull. Amer. Math. Soc. **53**, 116–117 (1947).

Let $P_n(z)$ denote the polynomial of degree n of best approximation to $f(z)$ in the closed region $|z|\leqq 1$, the radius of convergence of $f(z)=\sum a_n z^n$ being $\rho>1$. Walsh has established [Interpolation and Approximation in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 20, 1935, chap. 4] that

$$\limsup \{\max_{|z|\leqq 1} |f(z)-p_n(z)|\}^{1/n}=1/\rho.$$

In the present paper the authors show that a necessary and sufficient condition for

$$\lim \{\max_{|z|\leqq 1} |f(z)-p_n(z)|\}^{1/n}=1/\rho$$

is that $f(z)$ is not of lacunary structure.
S. Mandelbrojt (Houston, Tex.)

Walsh, J. L. Note on the derivatives of functions analytic in the unit circle. Bull. Amer. Math. Soc. **53**, 515–523 (1947).

L'auteur complète les résultats obtenus par W. Seidel et lui même [Trans. Amer. Math. Soc. **52**, 128–216 (1942); ces Rev. **4**, 215] concernant les rayons d'univalence $D_1(w_0)$ ou de p-valence $D_p(w_0)$ d'une fonction $f(z)$ de classe K_M, c'est-à-dire telle que $\iint |f'(z)|^2 ds\leqq \pi M^2$. Démonstration de l'inégalité $|f'(z_0)|(1-|z_0|^2)\leqq\{6MD_1(w_0)\}^{\frac{1}{2}}$ avec $w_0=f(z_0)$, et d'une inégalité analogue pour $D_p(0)$. Application à l'étude du comportement de $D_p(w_0)$ et de $f'(z_0)$ lorsque $|z_0|\to 1$.
J. Ferrand (Caen).

Dvoretzky, Aryeh. Les coefficients d'une fonction univalente et le domaine étalé. C. R. Acad. Sci. Paris **225**, 447–449 (1947).

Let $w=f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ be regular and univalent for $|z|<1$, mapping $|z|<1$ on a domain Δ of the w-plane. Let $\Delta(R)$ denote the radius of the largest circle having its center on $|w|=R$ and interior in Δ. It is well known that $a_n=O(n)$ and $A(R)=O(R)$ as $R\to\infty$. By restricting $A(R)=O(R^\alpha)$ the author obtains the following results: (1) if $0<\alpha\leqq 1$, then $a_n=O(n\alpha/(2-\alpha))$; (2) if $\alpha=0$, then $a_n=O(\log n)$; (3) if $\alpha<0$, then $a_n=o(1)$; (4) if $\alpha<-2$, then $a_n=O(n^{-1+2/(2-\alpha)})$. These results are readily obtained by means of the following theorem which the author states without proof. For $0<|z|=t<1$, $|f(z)|<R_t$, where R_t is the smallest positive root of the equation $A(R)=cR^2(1-t)^2/t$, c being a positive absolute constant. *M. S. Robertson.*

Alenitzyn, G. On mean p-valent functions. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 113–124 (1947). (Russian. English summary)

Suppose that $w=F(\zeta)$ is regular in $|\zeta|>1$ except for a pole at $\zeta=\infty$, and let W denote the Riemann domain which is the image of $|\zeta|>1$. Let $W(a;r,R)$ be the area (multiply-covered regions counted multiply) of that portion of W which lies in the ring $r<|w-a|<R$. In the cases $r=0$, $a=0$, $a=r=0$, let the corresponding areas be $W(a;R)$, $W(r,R)$, $W(R)$. The function $F(\zeta)$ is said to be mean p-valent (p a positive number) if there is a point a such that one of the following two conditions is satisfied: (a) $W(a;R)\leqq p\pi R^2$ for all $R>0$; (b) $W(a;r,R)\leqq p\pi(R^2-r^2)$ for any given $r\geqq 0$ and all sufficiently large R. The point a is called the center of W or of F; by making a translation we may suppose that $a=0$. When p is a positive integer and F is of the form $F(\zeta)=\zeta^p(1+\alpha_1/\zeta+\alpha_2/\zeta^2+\cdots)$, F is said to belong to the classes $\bar{\Sigma}_p$ or $\tilde{\Sigma}_p$ according as (a) or (b) is satisfied.

The author proves the following generalization of the area-principle as given by Prawitz [Ark. Mat. Astr. Fys. **20A**, no. 6 (1927)]. If $F=\zeta^p(1+\alpha_1/\zeta+\alpha_2/\zeta^2+\cdots)$ is mean p-valent with center zero in $|\zeta|>1$ (p a positive integer), then $(d/d\rho)\int_{-\pi}^{\pi}|F(\rho e^{i\varphi})|^\lambda d\varphi\geqq 0$ for $\rho>1$ provided that $0<\lambda\leqq 2$ if $F\epsilon\bar{\Sigma}_p$ and $\lambda\geqq 2$ if $F\epsilon\tilde{\Sigma}_p$. It is shown that the restrictions on λ are essential. [Generalizations of the area-principle to mean p-valent functions under somewhat different assumptions have been made by Biernacki, Bull. Sci. Math. (2) **70**, 51–76 (1946); these Rev. **8**, 326, and by the reviewer, Ann. of Math. (2) **42**, 614–633 (1941); these Rev. **3**, 78.] Writing $(F/\zeta^p)^{\lambda/2}=\sum_{-\infty}^{\infty}b_\nu\zeta^{-\nu}$, $b_0=1$, $b_1=\frac{1}{2}\lambda\alpha_1$, $\cdots$, an immediate consequence of this inequality is that $\sum_{\nu=1}^{\infty}(2\nu-p\lambda)|b_\nu|^2\leqq p\lambda$.

In addition, it is shown that if F belongs to $\bar{\Sigma}_p$ with center zero and $F\neq 0$ in $|\zeta|>1$, then the function $\{F(\zeta)\}^{1/p}$ belongs to $\bar{\Sigma}_1$, and $\{F(\zeta^n)\}^{1/n}$ (n a positive integer) belongs to $\bar{\Sigma}_p$, but that the corresponding statement for the class $\tilde{\Sigma}_p$ is generally false. *D. C. Spencer.*

***Caratheodory, Constantin. Probleme der analytischen Funktionen einer Veränderlichen.** Atti Convegno Mat. Roma 1942, pp. 209–213 (1945).

A lecture on problems connected with the behavior of a function near a singularity, and Bloch's constant.

***Ahlfors, L., et Beurling, A. Invariants conformes et problèmes extrémaux.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 341–351. Jul. Gjellerups Forlag, Copenhagen, 1947.

Two plane regions are said to belong to the same conformal type if they are conformally equivalent. More generally, one can consider configurations consisting of a plane region and certain preferred interior points, boundary points and boundary contours (understanding that this makes sense) and define two such configurations as belonging to the same conformal type provided that there exists a (1, 1) directly conformal mapping of one region onto the other which makes the preferred elements (points, contours) correspond. The present paper is concerned with the problem of determining a complete set of conformal invariants or modules for such configurations. Only special cases are considered, the general situation being reserved for later treatment. Use is made of the authors' theory of extremal length [an account of which will be published elsewhere]. The cases studied are (1) the quadrilateral (a simply-connected region with four preferred boundary points), (2) the pentagon (five preferred boundary points), (3) the hexagon (six preferred boundary points), (4) a triply-connected region. Case (4) is reduced to case (3). The pentagon and hexagon are treated in detail in terms of the theory of extremal lengths. It is shown in the case of the pentagon that the conformal types are in (1, 1) correspondence with the points of the first quadrant of the plane. A similar result holds for the hexagon. The proofs are carried through by conformal mapping onto canonical regions in which the extremal metric is Euclidean and a natural choice of modules lies at hand. *M. Heins* (Providence, R. I.).

Ahlfors, Lars V. Bounded analytic functions. Duke Math. J. **14**, 1–11 (1947).

The object of this paper is to treat the true equivalent of Schwarz's lemma for single-valued bounded analytic functions in a multiply-connected region. A related problem has been studied by Grunsky [cf. the following review for references] but his results do not include uniqueness considerations. A preliminary section is concerned with the relation between the so-called Painlevé problem of characterizing multiply-connected regions which tolerate non-constant bounded single-valued analytic functions, and the principal problem of the present paper. The author's method of attack on the main problem of the paper consists of the systematic use of variational methods (variation of zeros and boundary values subject to certain constraints corresponding to the single-valued character of the admitted functions). The author wishes to point out that there are lacunae in his argument [p. 5 leading to the statement that the number of finite zeros of an extremal is at most $n-1$; p. 7 leading to the statement (essentially) that the modulus of an extremal attains the value one continuously on the boundary], but that these difficulties can be overcome.

M. Heins (Providence, R. I.).

***Grunsky, Helmut. Zur Funktionentheorie in mehrfach zusammenhängenden Gebieten.** Ber. Math.-Tagung Tübingen 1946, pp. 68–69 (1947).

The author discusses briefly certain extremal problems for functions which are bounded and analytic in a given multiply-connected region of finite connectivity. These pertain to the author's earlier investigations [Jber. Deutsch. Math. Verein. **50**, 230–255 (1940); **52**, 118–132 (1942); these Rev. **2**, 275; **4**, 270] and to questions of the Pick-Nevanlinna type. No proofs are given. *M. Heins.*

Popovăţ, Petre. Sur les régions de monovalence des fonctions rationnelles. Disquisit. Math. Phys. **2**, 169–251 (1942).

L'auteur décompose le plan de la variable complexe x en régions de monovalence ou univalence pour un polynome $y=P(x)$ en utilisant le procédé classique qui utilise la surface de Riemann, mais en présentant les choses sous forme élémentaire. Il emploie une terminologie spéciale: il compte le nombre des points critiques de la fonction inverse sur le plan simple des y et néglige le point à l'infini. Il examine en détail trois cas particuliers correspondant au nombre maximum et minimum de points critiques d'une part, de sommets des régions d'univalence d'autre part. Ces cas sont d'une part le cas le plus général, et les cas qui se ramènent par transformation homographique à $y=x^n$ et à $y=T(x)$, $T(x)$ étant le polynome de la division des arcs (polynome de Tchebycheff). L'étude de $T(x)$ conduit l'auteur à établir, pour les polynomes, un théorème du genre Landau, mais les considérations de la page 207 semblent insuffisantes. Les fractions rationnelles $y=R(x)$ sont ensuite étudiées par les mêmes méthodes; les cas limites se ramènent à $y=x^n$ et aux fractions rationnelles qui s'introduisent dans la théorie des fonctions elliptiques lorsqu'on multiplie l'argument par un nombre rationnel. Enfin l'auteur est conduit à examiner le cas où il n'existe dans le plan des y que trois points qui sont projections de points critiques de la surface de Riemann de la fonction inverse, tous les points ainsi projetés étant effectivement critiques. Alors $R(x)$ vérifie une équation différentielle qui ne conduit pas à une solution explicite. L'auteur montre que les fonctions polyèdriques sont des cas particuliers de ces $R(x)$ et termine par les considérations sur les fonctions fuchsiennes de genre zéro.

G. Valiron (Paris).

Jongmans, F. Les transformations conformes non-biunivoques du plan. Bull. Soc. Roy. Sci. Liége **14**, 164–175 (1945).

Étant donnée une fraction rationnelle $Z=R(z)$, de degré n, l'auteur s'est proposé de partager le plan des z en domaines d'univalence. Mais, au lieu de chercher à former n domaines d'univalence complets, dans chacun desquels la fonction prendrait une fois et une seule toute valeur sauf celles correspondant à la frontière, il détermine $2n$ domaines dont le regroupement deux à deux redonne des domaines complets. A cet effet, il trace dans le plan des Z une courbe simple qui se ferme à distance finie ou à l'infini. Aux deux domaines limités par cette courbe dans le plan des Z correspondent $2n$ domaines d'univalence dans le plan des z. Quelques exemples d'application sont donnés. L'auteur apporte peu de faits nouveaux dans ce problème classique; les auteurs qu'il cite: Hibbert [Bull. Soc. Math. France **66**, 81–113, 115–154 (1938)], Marty [Ann. Fac. Sci. Univ. Toulouse (3) **23**, 183–261 (1931)] et Radoïtchitch [C. R. Acad. Sci. Paris **189**, 1240–1242 (1929); **190**, 356–357 (1930)], auxquels il faudrait ajouter notamment Iversen [Thèse, Helsingfors, 1914] et Shimizu [Jap. J. Math. **8**, 175–236, 237–304 (1931)] avaient surtout en vue le problème analogue, mais autrement difficile, relatif aux fonctions entières ou méromorphes. *G. Valiron* (Paris).

Carrier, G. F. On a conformal mapping technique. Quart. Appl. Math. **5**, 101–104 (1947).

The author describes a method of mapping a simply connected domain B into a circle so that two given interior points P and Q of B are mapped into points with affixes $+a$ and $-a$ inside the circle. This shows that the problem can be reduced to the determination of the Green's function of B. Using a method similar to that of Gerschgorin [Rec. Math. [Mat. Sbornik] **40**, 48–58 (1933); "Conformal Representation . . .," edited by Smirnov, Leningrad, 1937, pp. 121, 126], he derives an integral equation for the determination of the Green's function. He suggests solving of this integral equation by the relaxation method.

S. Bergman (Cambridge, Mass.).

Carrier, G. F. On the conformal mapping of airfoils. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 65–68 (1947). (English. Russian summary)

[In the original the author's initials were misprinted G. P.] The two dimensional potential flow past a grid of equal, parallel and equidistant aerofoils [cf. W. Traupel, Sulzer Tech. Rev. **1945**, no. 1, 25–42; these Rev. **7**, 423] is considered by methods of conformal representation and integral equations. [This is a summary of the paper reviewed above.] *A. J. Macintyre* (Aberdeen).

***Göllnitz, Erich. Über ganze Funktionen, die in den Ecken gewisser regulärer Polygone verschwinden.** Ber. Math.-Tagung Tübingen 1946, pp. 65–66 (1947).

The author observes that the entire function

$$G_r(z)=\prod_{n=1}^{\infty}\{1-(z/n)^r\},$$

for $r\geqq 2$, may be factored into $\prod_{\lambda=1}^{r}1/\Gamma(1-\omega^\lambda z)$, where ω is a primitive rth root of unity. If r is even, it can also be

factored as

$$-\prod_{\lambda=1}^{r/2}\{-(\pi z)^{-1}\sin \pi\omega^{2\lambda}z\}.$$

In the special case $r=6$, the author makes use of trigonometric identities to obtain

$$G_6(z)=6\sum{}'(-1)^{n+1}(2\pi z)^{2n-2}/(2n+1)!,$$

where the summation is restricted to $n\equiv 1 \pmod 3$. The same method also yields the identity

$$\prod_{n=0}^{\infty}\{1-(2z/2n+1)^6\}=1+\tfrac{3}{4}\sum_{n=1}^{\infty}(-1)^n(2\pi z)^{6n}/(6n)!.$$

R. C. Buck (Providence, R. I.).

Climescu, Al. C. Sur la classe des fonctions analytiques qui gardent les demi-plans déterminés par l'axe réel. Ann. Sci. Univ. Jassy. Sect. I. 28, 31–138 (1942).

Verf. behandelt im wesentlichen das folgende Problem. Es sind die Koeffizienten jener reellen Potenzreihen $f(z)=\sum_{n=0}^{\infty}a_nz^n$ mit dem Konvergenzradius 1 zu charakterisieren, welche die folgenden zwei Eigenschaften haben: (1) Die Funktion $f(z)$ ist in das Gebiet $\arg(z-1)\neq 0$ hinein analytisch fortsetzbar, (2) ihr Imaginärteil ist in der obern Halbebene positiv. Hinreichend für das genannte Verhalten ist der positiv definite Charakter der quadratischen Formen $\sum_{p,q=0}^{n}a_{p+q}x_px_q$ und $\sum_{p,q=0}^{n}a_{p+q+1}x_px_q$ für $n=0, 1, 2, \cdots$, notwendig dagegen ihr positiv semidefiniter Charakter. Von der Poisson-Stieltjes'schen Integraldarstellung für analytische Funktionen mit positivem Realteil gelangt Verf. über die typisch-reellen Funktionen von Rogosinski [Math. Z. 35, 93–121 (1932)] zur folgenden Integraldarstellung seiner Funktionen:

$$f(z)=a+bz+\int_0^1\frac{z\cdot d\phi(u)}{1-zu},$$

$\phi(u)$ monoton wachsend und beschränkt. Dies ergibt notwendige Bedingungen für die Koeffizienten. Die Lösbarkeitskriterien des Momentenproblems $a_n=\int_0^1 u^n\cdot d\phi(u)$ für monoton wachsendes und beschränktes ϕ liefern hinreichende Bedingungen. Einfacher als vom Verf. könnte hier so geschlossen werden: Aus dem positiv definiten Charakter der obigen Formen folgt zunächst die Lösbarkeit des Problems $a_n=\int_0^\infty u^n d\phi(u)$, ϕ monoton wachsend. Die weitere Bedingung $\limsup_{n\to\infty}|a_n|^{1/n}=1$ ergibt sofort, dass ϕ für $u>1$ konstant ist. Ähnlich sind Methode und Ergebnisse, wenn das Gebiet $\arg(z-1)\neq 0$ durch die von $z=1$ bis $z=-1$ über $z=\infty$ längs der reellen Achse aufgeschnittenen Ebene ersetzt wird. Mehrere damit zusammenhängende Fragen werden diskutiert. *A. Pfluger* (Zürich).

***Hasse, Helmut. Überblick über die neuere Entwickelung der arithmetischen Theorie der algebraischen Funktionen.** Atti Convegno Mat. Roma 1942, pp. 25–33 (1945).

***Fueter, Rudolph. Problèmes actuels de la théorie des fonctions analytiques de plusieurs variables.** Atti Convegno Mat. Roma 1942, pp. 169–177 (1945).

Rutishauser, H. Sur les suites et familles de fonctions méromorphes de plusieurs variables. C. R. Acad. Sci. Paris 224, 1804–1806 (1947).

Rutishauser, Heinz. Sur les suites et familles de représentations analytiques du R^4. C. R. Acad. Sci. Paris 225, 33–35 (1947).

Verf. nennt eine Folge $\mathfrak{F}$ von meromorphen Funktionen $f(w, z)$ quasi-regulär in einem Bereiche D, falls dort die singulären Punkte der Folge eine analytische Fläche V_2 bilden. Die Folge $\mathfrak{F}$ heisst in D irreduzibel, falls jede ihrer Teilfolgen ebenfalls auf ganz V_2 singulär ist. Eine Familie in D meromorpher Funktionen wird quasi-normal genannt, falls jede Funktionsfolge der Familie eine in D quasi-reguläre Teilfolge besitzt. Falls für alle irreduzibeln Folgen $\mathfrak{F}$ einer quasi-normalen Familie die zugehörigen Flächen V_2 nach oben beschränkte Ordnungen besitzen, so wird die obere Grenze s dieser Ordnungen auch die Ordnung der gegebenen Familie genannt. Verf. kündet dann einige Kriterien und Sätze über normale und quasi-normale Familien und quasi-reguläre Folgen an. Hierbei spielt das Verhalten der a-Stellenflächen der gegebenen Funktionen eine wesentliche Rolle. So ist eine in einem Bereiche D vorliegende Familie meromorpher Funktionen in D quasi-normal und dort von endlicher Ordnung, falls für drei von einander verschiedenen Konstanten a, b, c die Ordnung der a-, b- und c-Stellenflächen der Funktionen der Familie beschränkt ist. Ebenso hat die Beschränktheit der Anzahl der a-Stellen der Funktionen einer irreduzibeln meromorphen Folge auf gewissen parallelen Ebenenstücken in der Umgebung einer Teilfläche T der Fläche V_2 die Fortsetzbarkeit der Grenzfunktion auf T zur Folge. Schliesslich wird noch gefolgert, ebenfalls auf Grund der Anzahl der a-Stellen der Funktionen der gegebenen quasi-regulären Folge, dass man einer Komponente T der Fläche V_2, auf welcher die Grenzfunktion noch analytisch fortsetzbar ist, eine bestimmte Zahl n_T als Ordnung bezüglich der Folge $\mathfrak{F}$ zuordnen kann.

In der zweiten Arbeit erweitert Verf. die Begriffe und Ergebnisse der ersten auf Folgen und Familien analytischer Abbildungen im Raume R_4. Es werden reguläre und quasi-reguläre Abbildungsfolgen und normale und quasi-normale Familien meromorpher Abbildungen definiert und Kriterien für solche Familien aufgestellt. Insbesondere untersucht Verf. Familien und Folgen projektiver Abbildungen. Zunächst gilt, dass jede solche Familie im ganzen R_4 quasi-normal ist und dass, falls die irregulären Punkte überhaupt ein zweidimensionales Gebilde erfüllen, dieses notwendig eine analytische Ebene ist. Es werden dann noch die vier möglichen Typen irreduzibler Folgen projektiver Abbildungen angegeben und einige sich daraus ergebende Folgerungen. *P. Thullen* (Bogotá).

***Peschl, E. Über die Bilder von Sternbereichen. Ein allgemeiner Abbildungssatz im Raume mehrerer komplexer Veränderlichen.** Ber. Math.-Tagung Tübingen 1946, pp. 112–116 (1947).

The author considers the problem of analytically mapping a domain B in the space of several complex variables into a star domain, the mapping to be one-to-one. The pertinent definitions are given, and five theorems are stated, with an outline of the proof of the main theorem (theorem 4). A complete exposition is to appear elsewhere. If A is an inner mapping of a bounded domain with fixed point O, then it is known that the determinant Δ of the linear part of the development of A about O satisfies the inequality $|\Delta|\leqq 1$, with $\Delta=1$ holding if and only if A is an automorphism. The first theorem states that for such an inner mapping the characteristic roots ρ_j of the linear part satisfy the inequality $|\rho_j|\leqq 1$. The inner mapping is called a proper inner mapping if all ρ_j satisfy the inequality $|\rho_j|<1$. The second theorem states that, if A is a proper inner mapping of a domain B, then there exists a domain $R\subset B$ with $A(R)$ plus boundary interior to R, and for every domain D with D plus boundary interior to B, $A^m(D)\to 0$ as $m\to\infty$. The

third theorem states that if, for some domain $R \subset B$, $A^p(R)$ plus boundary is interior to R for some positive integer p, then A is a proper inner mapping of B. The author next defines infinitesimal inner mappings and in terms of these states his principal theorem. (This theorem, in contrast to the others, is obtained only for the case of two complex variables.) The theorem asserts that, if a domain B admits an infinitesimal proper inner mapping, then it is analytically mappable in a one-to-one manner either on a star-domain or a cyclic covering of such a domain. *W. T. Martin.*

*Rothstein, W. **Der Hartogssche Hauptsatz für reguläre une meromorphe Funktionen.** Ber. Math.-Tagung Tübingen 1946, pp. 125–126 (1947).

Verf. skizziert eine neue, sowohl auf reguläre wie meromorphe Funktionen anwendbare Beweismethode eines der grundlegenden Hartogsschen Sätze, der in der folgenden Formulierung angegeben wird: Sei $f(w, z)$ regulär (meromorph) in $|w|<1$, $|z|<1$, ferner jeweils bei festgehaltenem $w=w^*(|w^*|<1)$ $f(w^*, z)$ regulär (meromorph) in $|z|<2$. Dann ist $f(w, z)$ in $|w|<1$, $|z|<2$ regulär (meromorph). *P. Thullen* (Bogotá).

Spampinato, Nicolò. Sulla caratterizzazione geometrica delle varietà caratteristiche e pseudocaratteristiche dello spazio reale euclideo. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 255–262 (1940).

Aus der Inhaltsangabe des Verfassers: "Zunächst wird an die von Severi aufgestellten charakteristischen Eigenschaften geometrischer Art der analytischen und pseudoanalytischen Mannigfaltigkeiten erinnert und dann gezeigt, wie man mit Hilfe der Theorie der total analytischen Funktionen bi-komplexer Veränderlichen eine weitere geometrische charakteristische Eigenschaft jener Mannigfaltigkeiten gewinnen kann." *P. Thullen* (Bogotá).

Bencivenga, Ulderico. Sulla rappresentazione geometrica delle algebre doppie dotate di modulo. Atti Accad. Sci. Napoli (3) **2**, no. 7, 39 pp. (1946).

The two sets of matrices

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad \begin{pmatrix} x & y \\ 0 & x \end{pmatrix},$$

where x, y take all real values, define hypercomplex systems of rank 2 over the real number field known as the bireal and dual numbers, respectively. The purpose of the paper is to develop an elementary geometric theory of functions for each of these two hypercomplex systems analogous to the ordinary theory of functions for a complex variable.

Each of the two systems may be represented by points (x, y) in the plane. The modulus of a hypercomplex number is defined to be the positive square root of the absolute value of the determinant of the matrix which represents it. The argument of such a number is double the area enclosed by the x-axis (or the y-axis if the x-axis does not define a finite area), the line joining (x, y) to the origin and the curve of points of modulus 1 with some additional sign refinements. An analysis of the geometric equivalents of addition, multiplication, etc., of the dual and bireal numbers is given which is similar to that of the corresponding results for the complex numbers in the Gauss plane.

The author defines differentiability for the functions of the hypercomplex variables and discusses regions of convergence for their power series. Analogues are given of the Cauchy-Riemann equations, the Cauchy integral theorem and the conformality of mapping. *D. Derry.*

Theory of Series

Gustin, William. Gaussian means. Amer. Math. Monthly **54**, 332–335 (1947).

To a set A of n numbers $a>0$, associate another set $T \cdot A$ consisting of the values of the expression $(\sum q a^t)^{1/t}$ (suitably interpreted when $t=0$) for n fixed values of t and for fixed weights $q>0$, where $\sum q=1$; define by induction $T^k \cdot A = T \cdot T^{k-1} \cdot A$. Then, as $k \to \infty$, the n numbers of the set $T^k \cdot A$ tend to a same limit. *L. C. Young.*

Cattaneo, Paolo. Generalizzazione della successione di Fibonacci. Boll. Un. Mat. Ital. (3) **2**, 52–56 (1947).

Pettineo, B. Una classe di serie divergenti. Ann. Mat. Pura Appl. (4) **24**, 291–297 (1945).

Let s_n be the sequence of partial sums of a series $\sum u_n$ of positive terms. Let $\log^{(1)} x = \log x$, $\log^{(2)} x = \log\log x$, etc. Generalizing a theorem of Abel concerning series of the type $\sum u_n/s_n$, several results are given concerning the partial sums, convergence, and divergence of series of the type

$$\sum u_n/s_n \log s_n \log^{(2)} s_n \cdots \log^{(h_n)} s_n,$$

where $h_1, h_2, \cdots$ is a sequence of positive integers subject to the restriction that the numbers $\log^{(h_n)} s_n$ are positive. See a later paper by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 680–685 (1946); these Rev. **8**, 259]. *R. P. Agnew* (Ithaca, N. Y.).

Christov, Chr., and Tagamlizky, Y. On the convergence of integrals with an infinite upper limit of integration. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 289–310 (1946). (Bulgarian. English summary)

Part I [Christov]. Let $u(x)$ and $p(x)$ be positive bounded measurable functions, $x>0$, $\lim_{x\to\infty} u(x)=0$. The integral $\int_0^\infty u(x)^{p(x)}dx$ is finite for every $u(x)$ if and only if $c_n^{1/n}$ is bounded, where c_n is the measure of the set where $n-1<p(x)\leqq n$. Extension to the case where $p(x)$ is unbounded.

Part II [Tagamlizky]. The theorem of part I is generalized to the Lebesgue-Stieltjes integral $\int_0^\infty F(u(x), p(x))d\varphi(x)$, where $\varphi(x)$ is nondecreasing; $F(\xi, \eta)$ is continuous for $\xi \geqq 0$, $\eta > a$; $F(\xi, \eta)$ is an increasing function of ξ, and a decreasing function of η when $\xi<\delta$; $F(\xi, \infty)=0$ when $\xi<\delta$; $F(\xi, \eta)\neq 0$, $0<\xi<\delta$. Let $0<\alpha<\min(\delta, 1)$, $t_0>a$. Determine an increasing sequence $\{t_n\}$ so that $\alpha F(\alpha, t_n)=F(\alpha, t_{n+1})$. Let γ_n be the set of x for which $t_{n-1}<p(x)\leqq t_n$, $n>0$; $p(x)\leqq t_0$ for $n=0$. Then the integral is finite for every bounded Baire function $u(x)$ with $\lim_{x\to\infty} u(x)=0$ and a given Baire function $p(x)$, bounded in every finite interval and satisfying $p(x)>a$, if and only if $\sum \alpha^n \int_{\gamma_n} d\varphi(x)$ converges and $\int_{\gamma_0} F(\alpha, p(x))d\varphi(x)$ is finite. Corollary: under the same conditions there exists a positive α such that the integral converges if the condition $u(x)\to 0$ is replaced by $u(x)\leqq\alpha$ for large x. *R. P. Boas, Jr.* (Providence, R. I.).

Rajagopal, C. T. On Riesz summability and summability by Dirichlet's series. Amer. J. Math. **69**, 371–378 (1947).

The main result of the paper is the following theorem. Suppose that $F(t)=t\int_0^\infty A(u)e^{-ut}du$ converges for $t>0$, $\lim_{t\to+0} F(t)=s$, and that, for constants $k\geqq 0$, $K>0$, $xA_k(x)-A_{k+1}(x)\geqq -Kx^{k+1}$ $(x>0)$, where

$$A_0(x)=A(x), \quad A_r(x)=r\int_0^x (x-u)^{r-1}A(u)du, \quad r>0.$$

Then $A_{k+1}(x) \sim sx^{k+1}$ as $x \to \infty$. This extends the special case $k=1$ of the same theorem proved by Szász [Trans. Amer. Math. Soc. 39, 117–130 (1936)]. When $F(t)$ does not tend to a limit the author finds relationships between the upper and lower limits of $F(t)$ and $\sigma_k(x)$, thus extending to Laplace and Riesz transformations a result of Ramaswami [J. London Math. Soc. 10, 294–308 (1935)] on Abel and Cesàro transformations. *H. R. Pitt* (Belfast).

Kuttner, B. On positive Riesz and Abel typical means. Proc. London Math. Soc. (2) 49, 328–352 (1947).

An increasing divergent sequence μ_n and a positive order k determine the Riesz typical transformation (R, μ_n, k) by which a series $\sum a_n$ is summable to L if the transform

$$A(\mu_n, k, u) = (k!)^{-1} \sum_{\mu_n \leqq u} (u-\mu_n)^k a_n$$

converges to L as $u \to \infty$. It is shown that if $\lambda_n = \varphi(\mu_n)$, where $\varphi(u)$ is a nonnegative increasing function which tends to infinity less rapidly than u and which has an appropriate number of derivatives alternately nonnegative and nonpositive, then each real series $\sum a_n$ which makes $A(\mu_n, k, u)$ nonnegative must also make $A(\lambda_n, k, u)$ nonnegative. If, further, $\varphi''(u) > 0$, then positiveness of $A(\mu_n, k, u)$ implies that of $A(\lambda_n, k, u)$. The conditions are satisfied if $0 < r < 1$ and $\varphi(u) = u^r$ and if $\varphi(u) = \log u$.

An increasing divergent sequence μ_n determines a modified Abel (or Dirichlet-series) transformation $D(\mu_n)$ by which $\sum a_n$ is summable to L if the series in $f(\mu_n, s) = \sum_{n=0}^{\infty} a_n e^{-\mu_n s}$ converges when $s>0$ to a function $f(\mu_n, s)$ such that $f(\mu_n, s) \to L$ as $s \to 0$. If $\lambda_n = \varphi(\mu_n)$, where $\varphi(u)$ satisfies the same conditions noted above, and if the series for $f(\mu_n, s)$ and $f(\lambda_n, s)$ converge when $s>0$, then $f(\mu_n, s) \geqq 0$ implies $f(\lambda_n, s) \geqq 0$. It is shown that the last result implies Cartwright's theorem [Proc. London Math. Soc. (2) 31, 81–96 (1930)] that if $0 < r < 1$, $\varphi(u) = u^r$, $\lambda_n = \varphi(\mu_n)$ and $\sum a_n$ is summable $D(\mu_n)$ to L, then $\sum a_n$ is also summable $D(\lambda_n)$ to L provided the series for $f(\lambda_n, s)$ converges when $s>0$.

R. P. Agnew (Ithaca, N. Y.).

Obrechkoff, N. Sur l'équivalence des procédés *C*- et *H*- de sommation des séries divergentes. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 42, 97–144 (1946). (Bulgarian. French summary)

Let $k>0$, integer or not, and let $s_n^{(k)}$ and $h_n^{(k)}$ denote the Cesàro and Hölder transforms of order k of the sequence s_n of partial sums of a series $\sum u_n$. Let p_n be a sequence of nonzero numbers satisfying the conditions

$$\sum_{\nu=1}^{n} |p_\nu| = O(np_n), \quad \lim_{n\to\infty} (np_n)^{-1} \sum_{\nu=1}^{n} p_\nu = a \neq 0.$$

If $\lim s_n^{(k)}/p_n = g$, then

$$\lim h_n^{(k)}/p_n = [a^k \Gamma(a^{-1}+k)/\Gamma(k+1)\Gamma(a^{-1})]g;$$

and, conversely, the second equality implies the first.

This theorem, which yields the equivalence of the Cesàro and Hölder methods of summability when we set $p_n = 1$ for each n, was proved for the case in which k is a positive integer by I. Schur [Math. Z. 31, 391–407 (1929)]. The proof, pages 100–132, involves many lemmas.

The second theorem, proved in pages 133–142, involves Cesàro and Hölder integral transforms of order $p>0$ of functions $s(t)$ Lebesgue integrable over some interval $0 < t < a$. If one of the transforms

$$px^{-p} \int_0^x (x-t)^{p-1} s(t)\,dt, \quad \{x\Gamma(p)\}^{-1} \int_0^x \{\log(x/t)\}^{p-1} s(t)\,dt$$

has a limit as $x \to 0$, so also does the other and the limits are equal. The special cases in which p is a positive integer were proved by Hardy and Littlewood [Math. Z. 19, 67–96 (1923)]. The analogous problem for the case in which $x \to \infty$ has been studied by many authors; see Knopp [Math. Z. 47, 229–264 (1941); these Rev. 3, 296] and the bibliography of the subject given there. *R. P. Agnew* (Ithaca, N. Y.).

Wilansky, Albert. On the convergence of double series. Bull. Amer. Math. Soc. 53, 793–799 (1947).

Convergence of double series by the method of σ-sums [as used by the reviewer, Amer. Math. Monthly 52, 365–376 (1945); these Rev. 7, 13; see also these Rev. 7, 517 (review of article by L. Amerio)] is compared with so-called regular-convergence. (The series $\sum a_{ij}$ is regular-convergent if it is Pringsheim-convergent and if every row and column converges.) The author's findings: (i) regular-convergence is more powerful than σ-convergence (every series that is σ-convergent is also regular-convergent (to the same sum), but not conversely); (ii) regular-convergence is equivalent to the following modification of σ-convergence: let $k>1$ be an arbitrary (but fixed) integer; then allow only partial σ-sums that have not more than k "steps."

I. M. Sheffer (State College, Pa.).

Birindelli, C. Sopra recenti metodi di sommazione per le serie semplici estesi alle serie multiple. Portugaliae Math. 6, 1–32 (1947).

T. H. Gronwall [Ann. of Math. (2) 33, 101–117 (1932)] defined methods (f, g) of summability depending upon functions $f(z)$ and $g(z)$ having stated properties. He gave three fundamental theorems relating his methods to (1) the Euler-Abel power series method, (2) the Cesàro methods, and (3) each other. Birindelli gives extensions of these theorems to double series. *R. P. Agnew* (Ithaca, N. Y.).

***Lorentz, G. G. Beziehungen zwischen den Umkehrsätzen der Limitierungstheorie.** Ber. Math.-Tagung Tübingen 1946, pp. 97–99 (1947).

The method of summation A, applied to a series $\sum_{\nu=1}^{\infty} u_\nu$, is defined by the transformation $\sigma_m = \sum_{n=1}^{\infty} a_{mn} s_n$, $s_n = \sum_{\nu=1}^{n} u_\nu$, and is assumed to be consistent. If $\{n_k\}$ is a sequence of integers satisfying $0 < n_1 < n_2 < \cdots$ and $\{c_n\}$ a sequence of nonnegative numbers, the author considers results of the following types: (L) if $u_n = 0$ for $n \neq n_1, n_2, \cdots$, A-summability implies convergence; (T) if $u_n = o(c_n)$, A-summability implies convergence; (B) every bounded sequence $\{s_n\}$ with $s_n = 0$ for $n \neq n_k$ is A-summable if n_k increases sufficiently rapidly.

The conclusions are as follows. (1) If $\sum_{n_k < n < n_{k+1}} c_n = o(1)$, then (L) implies (T). (2) Theorem (B) is true if

$$\lim_{m\to\infty} \{ \max_{n=1,2,\cdots} |a_{mn}| \} = 0.$$

(3) The condition $u_n = O(c_n)$, together with Abel or Cesàro summability of positive order, ensures convergence if, and only if, for every $\epsilon > 0$ we can define integers $\{n_k\}$ to satisfy $n_{k+1}/n_k \geqq q > 1$ such that $\sum_{n_k < n < n_{k+1}} c_n < \epsilon$. *H. R. Pitt.*

Delange, Hubert. Théorèmes taubériens relatifs à l'intégrale de Laplace. C. R. Acad. Sci. Paris 224, 1802–1804 (1947).

The author proves the following results. Theorem 1. Suppose that $s(t)$ is a real or complex function of t for $t>0$

and that for some $\lambda>1$ and $\alpha\geqq 0$ we have

(1) $$\overline{\lim_{t\to\infty}}\ \text{u.b.}_{t\leq t'<\lambda t}\ t^{-\alpha}|s(t')-s(t)|<\infty.$$

Suppose also that the complex sequence $z_n=r_ne^{i\theta_n}$ satisfies

(2) $$|\theta_n|<\pi/2,\quad \overline{\lim_{n\to\infty}}\,|\theta_n|<\pi/2,\quad \lim_{n\to\infty} r_{n+1}/r_n=1,$$

and that

$$z_n\int_0^\infty e^{-z_nt}s(t)dt=Cz_n^{-\alpha}+o(r_n^{-\alpha}).$$

Then if E is any unbounded set of positive numbers,

(3) $$\overline{\lim_{t\to\infty,\ t\in E}}\left|\frac{s(t)}{t^\alpha}-\frac{C}{\Gamma(\alpha+1)}\right| \leqq \lim_{\lambda\to 1+0}\ \overline{\lim_{t\to\infty,\ t\in E}}\ \text{u.b.}_{t\leq t'<\lambda t}\ t^{-\alpha}|s(t')-s(t)|.$$

Theorem 2. If conditions (1) and (2) of theorem 1 hold with $-(p+1)<\alpha<-p$ for a nonnegative integer p, and if for some constants $C, C_0, C_1, \cdots, C_p$, we have

$$z_n\int_0^\infty e^{-z_nt}s(t)dt=C_0+C_1z_n+\cdots+C_pz_n^p-Cz_n^{-\alpha}+o(r_n^{-\alpha}),$$

then $s(+\infty)=C_0$ and

$$\int_0^\infty [s(+\infty)-s(t)]t^{q-1}dt=(-1)^q q!C_q$$

for $q=1, 2, \cdots, p$. Also, for any set E, the inequality (3) of theorem 1 remains true if we replace $s(t)$ by $s(+\infty)-s(t)$. There are analogous "one-sided" theorems when $s(t)$ is real.

H. R. Pitt (Belfast).

Delange, Hubert. Théorèmes taubériens généraux. C. R. Acad. Sci. Paris **225**, 28–31 (1947).

The author proves the following Tauberian theorem, with an analogous "one-sided" theorem when the functions $N(u)$ and $s(t)$ are real. Theorem. Suppose that $N(u)$ and $N(u)\log u$ are integrable in $(0, \infty)$, that $\int_0^\infty N(u)du=1$ and that $\int_0^\infty N(t)t^{iu}dt\neq 0$ for any real u. Suppose that

(1) $$\lim_{t\to\infty}\ \text{u.b.}_{t\leq t'<\lambda t}\ |s(t')-s(t)|<\infty$$

for some $\lambda>1$ and that for every real k

$$\Phi(kx)-\Phi(x)=(\Phi(x)/\log x)\log k+o(1)$$

as $x\to\infty$, where

$$\Phi(x)=(1/x)\int_0^\infty N(t/x)s(t)dt.$$

Then if E is any unbounded set of positive numbers,

$$\overline{\lim_{t\to\infty,\ t\in E}}\ |s(t)-\Phi(t)-C\Phi(t)/\log t|\leqq \lim_{\lambda\to 1+0}\ \overline{\lim_{t\to\infty,\ t\in E}}\ t^{-\alpha}|s(t')-s(t)|,$$

where

$$C=\int_0^\infty N(u)\log(1/u)\,du.$$

H. R. Pitt (Belfast).

Agnew, R. P., and Walker, R. J. A trigonometric infinite product. Amer. Math. Monthly **54**, 206–211 (1947).

Let A denote the set of real numbers x not of the form $h\pi/2^k$, where h is an odd and k a positive integer, and let $P_n(x)$ be the product

$$(\tan x)(\tan 2x)^{\frac{1}{2}}(\tan 4x)^{\frac{1}{4}}\cdots(\tan 2^nx)^{(\frac{1}{2})^n}.$$

For x in A Dobinski [Arch. Math. Phys. **59**, 98–100 (1876)] gave two "proofs" of the formula $\lim_{n\to\infty}P_n(x)=4\sin^2 x$. Noting that his first proof breaks down because of an unjustifiable interchange of limit processes, the authors examine the conditions under which the formula holds. It is shown that the formula is valid only in trivial cases unless the values of $(\tan 2^nx)^{(\frac{1}{2})^n}$ are suitably chosen complex numbers. By expressing x in dyadic form and applying a theorem of Borel [Rend. Circ. Mat. Palermo **27**, 247–271 (1909)], the authors prove that for such choices Dobinski's formula is valid for each x in the interval $0<x<\pi/2$ except for a set having Lebesgue measure zero. Finally, if x denotes any complex number and θ any real number, the authors prove that the values of $(\tan 2^nx)^{(\frac{1}{2})^n}$ for successive integral values of n can be chosen so that $\lim_{n\to\infty}P_n(x)=4e^{2I(x)+\theta}\sin^2 x$, provided that x does not belong to a certain set of measure zero on the real axis.

W. H. Gage (Vancouver, B. C.).

Scott, W. T., and Wall, H. S. On the convergence and divergence of continued fractions. Amer. J. Math. **69**, 551–561 (1947).

The authors derive conditions which are necessary and conditions which are sufficient for the convergence of the continued fraction $\mathrm{K}(1/b_n)$; they include many well-known results as special cases.

O. Szász (Cincinnati, Ohio).

Fourier Series and Generalizations, Integral Transforms

Cheng, Min-Teh. Summability factors of Fourier series at a given point. Duke Math. J. **14**, 405–410 (1947).

Let $f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^\infty(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^\infty A_n(x)$. At any point x at which $\int_0^t|\varphi_x(u)|\,du=O(t(\log 1/t)^\beta)$ $(\beta\geqq 0)$ holds, the series $\sum_{n=2}^\infty A_n(x)\lambda(n)$ is summable as follows:

(i) $|C,\alpha|$ with $\alpha>0$, for $\lambda(n)=(\log n)^{-(1+\beta+\epsilon)}$ $(\epsilon>0)$,
(ii) $|C,1|$ for $\lambda(n)=(\log n)^{-\beta-\epsilon-1}$ $(\epsilon>0)$,
(iii) $|C,\gamma|$ with $\gamma>\alpha$, for $\lambda(n)=n^{\alpha-1}(\log n)^{-\beta}$ $(0<\alpha<1)$,
(iv) $|C,\alpha|$ with $0\leqq\alpha<1$, for $\lambda(n)=n^{\alpha-1}(\log n)^{-(1+\beta+\epsilon)}$ $(\epsilon>0)$.

P. Civin (Eugene, Ore.).

Cheng, Min-Teh. Cesàro summability of orthogonal series. Duke Math. J. **14**, 401–404 (1947).

If $\{A_n\}$ is a sequence of real numbers and $\{\varphi_n(x)\}$ is a normal orthogonal set of functions defined in (a, b) such that $|\varphi_n(x)|<A$ and $\int_a^b\varphi_n(x)dx=0$ for $n=1, 2, 3, \cdots$, A being a constant, then the condition $\sum_{n=1}^\infty A_n^2n^k<\infty$, $1<k<2$, implies that the series $\sum_{n=1}^\infty A_n\varphi_n(x)$ is summable $(C, -k/2)$ for almost every x. This is a generalization of a result on trigonometrical series due to M. Jacob [Proc. London Math. Soc. (2) **26**, 470–492 (1927)] who, however, proved it for $0<k<2$.

K. Chandrasekharan.

Timan, A. Sur une méthode d'approximation des fonctions continues au moyen des polynomes trigonométriques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **11**, 263–282 (1947). (Russian. French summary)

Let $S_n(x)=S_n(x, f)$ denote the partial sums of the Fourier series of a function $f(x)$. Rogosinski proved [Math. Ann. **95**, 110–134 (1925)] that, if f is continuous and k any fixed

odd integer, then

(*) $\omega_n(k;f,x)=\frac{1}{2}\{S_n(x+k\pi/(2n+1))+S_n(x-k\pi/(2n+1))\}$

converges uniformly to $f(x)$. The author considers the behavior of (*) under the assumption that $k=k(n)$ varies with n and proves that a necessary and sufficient condition that (*) converges uniformly to $f(x)$ for every continuous and periodic f is that $k(n)$ is odd for all n large enough and that $k(n)=O(1)$ [the sufficiency here is a consequence of Rogosinski's theorem.] Parallel results are obtained for expressions obtained from (*) by replacing there the sum of the S_n by their difference and for trigonometric interpolation with equidistant fundamental points. *A. Zygmund.*

Korevaar, J. The uniform approximation to continuous functions by linear aggregates of functions of a given set. Duke Math. J. 14, 31–50 (1947).

A number of closure theorems are proved, of which the following are typical. (I) Let $\{\phi_n(x)\}$ be a closed set in $C[a,b]$, and let $h(x)$ be a continuous function which vanishes at k points of the interval. Then the set

$$\{1, x, \cdots, x^{k-1}, h(x)\phi_n(x)\}$$

is closed. (II) The set $\{1, x, (1-x)^\alpha x^n/(1-x^n)\}_1^\infty$ is closed $C[0,1]$ when $\alpha\geqq 1$. This completes and corrects a result of Hille and Szász [Bull. Amer. Math. Soc. 42, 411–418 (1936)]. (III) The set $\{x^{k-1}/(1+x^k)\}_1^\infty$ is not closed $C[0,1]$. (IV) Let $C_0[0,1]$ denote the set of continuous functions on $[0,1]$, vanishing at the end points. Let $\{s_n\}$ be a sequence of positive numbers having a limit point in $0<x<\infty$. Then $\{f(x^{s_n})\}$ is closed $C_0[0,1]$ provided that $f(z)$ is analytic in $|z|<1$ and at $z=1$, and not constant. (V) Under the preceding hypotheses on $f(z)$, and the additional condition $|f(z)|\leqq M$, the set $\{f(x^n)\}_1^\infty$ is closed $C_0[0,1]$.

H. Pollard (Ithaca, N. Y.).

Landkof, N. Sur la densité de certains systèmes de fonctions harmoniques dans l'espace des fonctions continues sur un ensemble. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 7–8 (1947).

Let E be a closed and bounded set of points in the complex z-plane, such that E has no interior points. Let $C(E)$ be the normed linear space of all real functions $f(z)$ continuous on E, with norm defined by $\|f\|=\max|f(z)|$ for $z\varepsilon E$. Let Ω be the complement of E, so that $\Omega=\sum_{k=0}^\infty \Omega_k$, where the Ω_k are domains. Let $\{O_k\}$ be a partial sequence of $\{\Omega_k\}$, with frontiers r_k, such that (*) $E=\bar{r}$, $r=\sum_{k=0}^\infty r_k$, but such that (*) is not satisfied if any r_k is omitted. Let z_k be a fixed point of O_k and let $\lambda(E)$ be the real linear envelope of the functions $\Re 1/(z-z_k)^n$, $\Im 1/(z-z_k)^n$, and $\log 1/|z-z_k|$, where k and n take on all nonnegative integral values; but $\log 1/|z-z_k|$ is omitted if z_k is in an infinite domain.

The author seeks conditions under which $\lambda(E)$ is dense in the space $C(E)$. The two following results are announced. In order that $\lambda(E)$ be dense in $C(E)$, it is sufficient that E contain no indecomposable continuum. The class $\lambda(E)$ is dense in $C(E)$ if and only if the open set $\sum_{k=0}^\infty O_k$ has no irregular point. *E. F. Beckenbach* (Los Angeles, Calif.).

Doetsch, Gustav. Das Verhalten der Laplace-Transformierten in ihrer Beschränktheitshalbebene. Comment. Math. Helv. 20, 1–8 (1947).

Let $f(s)$ be the Laplace transform of $F(t)$ convergent for $\Re(s)>\beta$ and suppose that $f(s)$ is holomorphic in $\Re(s)>\eta$ where $\eta<\beta$. Let $\mu=\inf\sigma$, $\sigma>\eta$, such that $|f(s)|\leqq M_\sigma$ when $\Re(s)\geqq\sigma$; μ is called the abscissa of boundedness, $\Re(s)>\mu$ the halfplane of boundedness of $f(s)$. Let

$$M(x)=\sup_y |f(x+iy)|$$

for fixed $x>\mu$. Here $M(x)$ is monotone decreasing and $\log M(x)$ is convex. Assuming that $f(s)$ has a half-plane of boundedness, the author proves

(1) $$\lim_{x\to\infty} x^{-1}\log M(x)=-v\leqq 0$$

exists, (2) $\limsup_{x\to\infty} x^{-1}\log|f(x)|=-v$, and (3) $F(t)\sim 0$ in $(0,v)$ and in no longer interval. He observes that if the left member of (2) equals $-\infty$, $f(s)$ being a Laplace transform implies $f(s)\equiv 0$. A couple of unsolved problems, mentioned by the author, are also of interest: (i) can $\mu<\beta$? (ii) can $\lim_{x\to\infty} M(x)$ be positive? In connection with (i) the author mentions a couple of unpublished results. In particular, the (C,ϵ) means of the integral must converge to $f(s)$ for $\epsilon>0$ and $\Re(s)>\mu$. [Reviewer's remark. For the case of an ordinary Dirichlet series, that is, the Laplace-Stieltjes transform of a special step function, μ equals the abscissa of uniform convergence as shown by H. Bohr. The example in footnote (1) of a Laplace transform for which $\eta<\beta$ also occurs on p. 58 of D. V. Widder, The Laplace Transform, Princeton University Press, 1941; these Rev. 3, 232.]

E. Hille (New Haven, Conn.).

Lebedev, N. N. Sur un formule d'inversion. C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 655–658 (1946).

Let $K_\nu(t)$ denote Macdonald's Bessel function [cf. Watson, Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922, p. 78]. The author proves the following formula:

$$xf(x)=2\pi^{-2}\int_0^\infty K_{i\tau}(x)\tau\sinh\pi\tau d\tau\int_0^\infty K_{i\tau}(\xi)f(\xi)d\xi,$$

subject to the (sufficient) conditions that $x^2 f(x)$ and $xf'(x)$ are integrable-L in $(0,\infty)$. *A. C. Offord.*

Beurling, Arne. Sur la composition d'une fonction sommable et d'une fonction bornée. C. R. Acad. Sci. Paris 225, 274–275 (1947).

Let K be integrable, ϕ bounded and measurable, and ψ bounded and continuous on $(-\infty,\infty)$, and suppose that $\int_{-\infty}^\infty |y|^{\frac{1}{2}}|K(y)|dy<\infty$. Define

$$U_\phi(\sigma,t)=\int_{-\infty}^\infty \phi(y)e^{-it y-\sigma|y|}dy, \qquad \sigma>0.$$

Theorem: in order that $\psi=K*\phi$ it is necessary and sufficient that

$$(2\pi)^{-1}\int_{-\infty}^\infty |k(t)U_\phi(\sigma,t)-U_\psi(\sigma,t)|^2 dt\to 0, \quad \sigma\to 0+,$$

where $k(t)$ is the Fourier transform of K. *H. Pollard.*

Beurling, Arne. Sur une classe de fonctions presque-périodiques. C. R. Acad. Sci. Paris 225, 326–328 (1947).

The following theorem is proved. Let $K(x)$ be an integrable function whose Fourier transform vanishes on at most a denumerable set of points λ_n without a finite limit point. Then any bounded uniformly continuous solution of $\int_{-\infty}^\infty K(y)\phi(x-y)dy=0$ is almost-periodic (Bohr) with frequencies $\{\lambda_n\}$. A similar theorem was proved earlier by Lewitan, but is overlooked by the author. [See Bull. Amer. Math. Soc. 43, 677–679 (1937).] *H. Pollard.*

Polynomials, Polynomial Approximations

***Baier, O. Die Hurwitzschen Bedingungen.** Ber. Math.-Tagung Tübingen 1946, pp. 40–41 (1947).

The author sketches a proof of the Hurwitz criterion for all the zeros of a real polynomial $f(z)$ to have negative real parts. His proof uses the fact that the zeros of $R(y)=\Re f(iy)$ and $I(z)=\Im f(iy)$ are all real and separate one another. If the negative-remainder division algorithm is applied to the polynomials $R_1(t)=R(t^2)$ and $I_1(t)=-I(t^2)/t^2$, the resulting Sturm sequence has certain positive coefficients which turn out to be the Hurwitz determinants. [Reviewer's note: A similar proof is given by H. S. Wall, Amer. Math. Monthly 52, 308–322 (1945); these Rev. 7, 62.]

M. Marden (Milwaukee, Wis.).

de Bruijn, N. G., and Springer, T. A. On the zeros of a polynomial and of its derivative. II. Nederl. Akad. Wetensch., Proc. 50, 264–270 = Indagationes Math. 9, 458–464 (1947).

The authors prove among others the following theorems. Let $f(z)$ be a polynomial of degree n with roots $y_1, \cdots, y_n$; the roots of $f'(z)$ are $z_1, \cdots, z_{n-1}$. Then

$$(1)\qquad n^{-1}\sum_{\nu=1}^{n}|\Im(y_\nu)| \geqq (n-1)^{-1}\sum_{\nu=1}^{n-1}|\Im(z_\nu)|,$$

$$(2)\qquad n^{-1}\sum_{\nu=1}^{n}|y_\nu| \geqq (n-1)^{-1}\sum_{\nu=1}^{n-1}|z_\nu|.$$

If all the coefficients of $f(z)$ are real de Bruijn proved (1) in a previous paper [same Proc. 49, 1037–1044 = Indagationes Math. 8, 635–642 (1946); these Rev. 8, 377].

P. Erdös (Syracuse, N. Y.).

Merli, Luigi. La convergenza in media del quarto ordine dei polinomi di interpolazione di Lagrange relativi ad un particolare sistema di punti interpolanti. Ann. Mat. Pura Appl. (4) 24, 283–289 (1945).

Let $f(x)$ be a continuous function defined for $-1\leqq x\leqq 1$ and satisfying a Lipschitz condition of order greater than $\frac{1}{4}$. Let $\{L_n(x)\}$ be the sequence of the Lagrange interpolating polynomials coinciding with $f(x)$ at the zeros of the Jacobi polynomials $I_n(\frac{1}{2}, -\frac{1}{2}, x)$ $(=\sin(n+\frac{1}{2})\theta/\sin\frac{1}{2}\theta$, if $x=\cos\theta)$. It is shown that then

$$\lim_{n\to\infty}\int_{-1}^{+1}(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}\{f(x)-L_n(x)\}^2dx=0.$$

A. Zygmund (Chicago, Ill.).

Shohat, J. A. Orthogonal polynomials and polynomials in general. Math. Mag. 21, 3–21 (1947).

An expository article.

Geronimus, J. The orthogonality of some systems of polynomials. Duke Math. J. 14, 503–510 (1947).

The author gives conditions for the orthogonality of systems of polynomials with respect to suitable distribution functions. In this way he brings under a single head a large class of the familiar polynomials (Appell, Stieltjes, Laguerre, Hermite, among others). *H. Pollard.*

Cotlar, Mischa. Study of a class of Bernoulli polynomials. Math. Notae 6, 69–95 (1946). (Spanish)

The author studies some of the properties of polynomials $g_n(x)$ which may be defined as the numerators of the nth reduced convergents of the continued fraction

$$x+\cfrac{1}{(x+1)+\cfrac{2}{(x+2)+\cdots+\cfrac{n-1}{(x-n+1)+\cdots}}},$$

and satisfy the recurrence

$$g_n(x)=(n-1+x)g_{n-1}(x)+(n-1)g_{n-2}(x).$$

For r a positive integer, $g_n(r)=f_n r$ and $g_n(0)=f_n$ constitute a set of numbers previously studied by the author [Math. Notae 5, 89–107 (1945); these Rev. 7, 242]. In the present paper the author considers the behavior of $g_n(r)$ for r a negative integer. An important property of these numbers is that of the permutability of the index n and the argument r.

The author also studies the sequence $\{M_n(x)\}$, where

$$M_n(x)=\sum_{i=0}^{n}(2^i/i!)C_n^i\sigma_i(x),$$

$$\sigma_i(x)=x(x-1)(x-2)\cdots(x-i+1).$$

It is shown that these polynomials and the polynomials $g_n(x)$ are the only Bernoulli polynomials which have the property of permutability referred to above. The paper contains a brief summary of the basic properties of the Bernoulli polynomials [reference being made to Nielsen's "Traité Élémentaire des Nombres de Bernoulli," Gauthier-Villars, Paris, 1923, and to R. Lagrange, Acta Math. 51, 201–309 (1928)]. *M. A. Basoco* (Lincoln, Neb.).

Special Functions

Kuznetsov, P. I. An expression of a contour integral. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 267–270 (1947). (Russian. English summary)

The author expresses by means of Lommel and Bessel functions $\int_C F(z, R)\exp(a+bz+R)dz$, where R is a square root of a polynomial of 2d degree in z, F is a rational function of z and R, a and b are constants and C is a contour enclosing all the singularities of the integrand. The result is applied to the propagation of an electromagnetic wave along a linear semiinfinite conductor. *I. Opatowski.*

Benham, T. A. Bessel functions in physics: theory. Amer. J. Phys. 15, 285–294 (1947).

Baudoux, P. Sur les fonctions de Struve. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 32 (1946), 127–131 (1947).

The Struve function $S_n(x)$ is a particular solution of the differential equation

$$y''+x^{-1}y'+(1-n^2/x^2)y=\frac{(\frac{1}{2}x)^{n-1}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}.$$

The author obtains the Laplace transforms of $t^nS_n(t)$ and of $t^{\frac{1}{2}n}S_n(2\sqrt{t})$ directly from the differential equation, and uses the latter Laplace transform to establish the expansion

$$S_n(x)=2\pi^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}x)^{k+1}}{k!(2k+1)}J_{n+k+\frac{1}{2}}(x).$$

A. Erdélyi (Pasadena, Calif.).

Baudoux, P. Sur quelques solutions particulières d'équations différentielles linéaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 32 (1946), 132–139 (1947).

The author gives a few properties of the inverse Laplace transforms of $p^{-1}L_n(p^{-1})$, $p^{-1}H_n(p^{-1})$, $p^{-1}J_n(p^{-1})$, $p^{-1}P_n(p^{-1})$.

A. Erdélyi (Pasadena, Calif.).

Magnus, Wilhelm. Über eine Bezeihung zwischen Whittakerschen Funktionen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. 1946, 4–5 (1946).

Proof of the addition theorem

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi(\kappa,\mu;z)\psi(-\kappa,\mu;\xi)d\kappa=\psi(0,2\mu+\tfrac{1}{2};z+\xi),$$

in which $\psi(\kappa,\mu;z)=\Gamma(\mu+\frac{1}{2}-\kappa)z^{-\mu-\frac{1}{2}}e^{\frac{1}{2}z}W_{\kappa,\mu}(z)$ and W is Whittaker's confluent hypergeometric function.

A. Erdélyi (Pasadena, Calif.).

***McLachlan, N. W. Theory and Application of Mathieu Functions.** Oxford, at the Clarenden Press, 1947. xii+401 pp. $12.50.

This book fills a long felt gap in the present literature on Mathieu functions. The contents are divided into two parts: theory of Mathieu functions and applications of Mathieu functions. The first part proceeds mainly along the method of Fourier series and continued fractions, which was so successfully applied by E. L. Ince. The Fourier forms as well as the Bessel forms of the solutions are dealt with elaborately. An excursion is made into the theory of Hill's general equation with bounded coefficients. The integral equations and the interrelations of the solutions are dealt with. Asymptotic solutions are discussed at length. Finally several useful integrals involving Mathieu functions are considered. The applications cover most of what is known today. The treatment is brief and adequate, numerous diagrams illustrating the material. A list of 226 references supplies useful additional material for students. A number of additional results, obtained by the author after completion of the book, are given at the end. *M. J. O. Strutt.*

McLachlan, N. W. Computation of the solutions of $(1+2\epsilon\cos 2z)y''+\theta y=0$; frequency modulation functions. J. Appl. Phys. 18, 723–731 (1947).

Solutions of the equation of the title are written as infinite Fourier series containing a phase constant. Recurrence relations for the coefficients of the four essentially different Fourier series are then obtained. Then the convergence of these series is discussed. The calculation of the phase constant is carried out by a trial method which is illustrated by some examples. A knowledge of this constant directly entails values for the coefficients of the series solutions. Several features and differences of the separate solutions are discussed. Finally the author deals with the case in which the coefficient m is large. The normalization of the solutions obtained calls for some separate considerations.

M. J. O. Strutt (Eindhoven).

Campbell, Robert. Sur une forme des fonctions de Mathieu (et de Mathieu associées) de periode $2s\pi$. C. R. Acad. Sci. Paris 224, 1322–1324 (1947).

The author applies to the associated Mathieu equation

$$d^2y/dx^2+2\nu\cot x\,dy/dx+(a+k^2\cos^2 x)y=0$$

the transformation $\cos x=z$ and obtains

$$(1-z^2)d^2y/dz^2+(2\nu+1)z\,dy/dz+(a+k^2z^2)y=0.$$

This equation has the two regular singular points -1 and 1 and if ν is different from half an odd integer it has two independent solutions in the form of power series. A solution of period $2s\pi$ of the original equation occurs if a solution of the transformed equation returns to its starting value after s encirclings of both singular points. From this property values of the characteristic multipliers may be obtained by a determinantal equation. The author obtains explicit expressions for these multipliers and thus obtains a transcendental equation relating these parameters in the case of a solution of the given periodicity. Finally he gives general expressions for the corresponding solutions themselves. *M. J. O. Strutt* (Eindhoven).

Harmonic Functions, Potential Theory

Zagar, Francesco. Attrazione e potenziale di ellissoidi. Rend. Sem. Mat. Univ. Padova 13, 57–77 (1942).

The potential of a homogeneous general ellipsoid is developed in a power series up to the sixth powers of the two eccentricities; the mutual potential of two ellipsoids with one axis parallel is developed up to the fourth powers.

F. Zernike (Baltimore, Md.).

Lebedev, N. N. Solution of Dirichlet's problem for hyperboloids of revolution. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 251–258 (1947). (Russian. English summary)

This paper deals with boundary value problems that can be solved by the use of Mehler's transformation defined by

$$\text{(I)}\qquad \Phi(\tau)=\int_1^\infty f(\xi)P_{-\frac{1}{2}+i\tau}(\xi)d\xi,\qquad 0\leqq\tau<\infty,$$

where $P_n(\xi)$ is Legendre's function of degree n. A typical case considered is Dirichlet's problem for hyperboloids of revolution. Through the application of the transformation (I) to Laplace's equation the problem is reduced to the solution of an ordinary differential equation, which is obtained in terms of Legendre's functions of complex degree. The solution of the original problem is then found by the inversion of the integral in (I) by means of the known formula

$$f(\xi)=\int_0^\infty \tau\tanh\tau\pi P_{-\frac{1}{2}+i\tau}(\xi)\Phi(\tau)d\tau$$

[F. G. Mehler, Math. Ann. 18, 161–194 (1881); V. A. Fock, C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 253–256 (1943); these Rev. 5, 181]. *H. P. Thielman* (Ames, Iowa).

Landkoff, N. Sur la résolubilité du problème de Dirichlet généralisé. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 11, 181–196 (1947). (Russian. French summary)

Appelons, dans le problème de Dirichlet (pour le domaine Ω de frontière F), ensemble résolvant E un ensemble de points irréguliers tel que pour tout φ bornée continue (sur F), la solution généralisée H_φ tende en tout $P\in F$ vers $\varphi(P)$ s'il en est ainsi aux points de E. Keldych avait annoncé l'existence d'un ensemble résolvant dénombrable [C. R. (Doklady) Acad. Sci. URSS (N.S.) 18, 315–318 (1938)]. L'auteur le démontre à partir d'un résultat étendant un peu un théorème de Frostmann: si $P_n\in\Omega\cup F$ tend vers Q irrégulier et si la distribution des "masses de Green" μ_{P_n} (obtenue par balayage généralisé) converge (vaguement au sens de H. Cartan) vers $\nu(e)$, alors ν vaut la réunion d'une masse ponctuelle m en Q et de $(1-m)\mu_Q$. Allant plus loin l'auteur montre que parmi les ensembles dénombrables résolvants, il y en a un minimum, formé des Q irréguliers jouissant de la propriété que μ_P tend vers la masse-unité en Q quand P irrégulier ($P\neq Q$) tend vers Q. Cette convergence de μ_P

quand P décrit seulement un suite $P_n \to Q$ équivaut à ce que H_φ ne tend pas toujours vers $\varphi(Q)$ quand H_φ prend les valeurs $\varphi(P_n)$ aux points P_n (c'est-à-dire tend vers $\varphi(P_n)$ en tout P_n). *M. Brelot* (Grenoble).

*Maak, W. **Zur Theorie der Kugelfunktionen.** Ber. Math.-Tagung Tübingen 1946, pp. 99–100 (1947).

The author states the familiar fact that harmonics on the sphere are the almost periodic functions corresponding to the group of orthogonal transformations acting transitively on the sphere. *S. Bochner* (Princeton, N. J.).

*Maruhn, Karl. **Über das Verhalten der Potentialfunktionen im Unendlichen.** Ber. Math.-Tagung Tübingen 1946, pp. 106–108 (1947).

In a three dimensional space let T_a be the exterior region bounded by a surface S. It is assumed that S is bounded and sufficiently smooth to permit the use of Green's identities. Let u be a potential function regular at each finite point of T_a and continuous together with its normal derivative on S, and further such that $\lim_{R\to\infty} D_n u = 0$, where $R=(x^2+y^2+z^2)^{\frac{1}{2}}$ and $D_n u$ is any nth order partial derivative of u. The author shows that

$$u = V + MR^{-1} + C + \sum_{k=1}^{n-1} \{c_1^{(k)} H_1^{(k)} + \cdots + c_{2k+1}^{(k)} H_{2k+1}^{(k)}\}.$$

Here V is a potential function regular in T_a; $V=O(R^{-3})$ for $R\to\infty$. The functions $H_1^{(k)}, \cdots, H_{2k+1}^{(k)}$ are linearly independent spherical harmonics of order k, the $c_m^{(k)}$ and C are arbitrary constants and M is a constant determined by u. The author states that similar results are valid for logarithmic potentials. *F. W. Perkins* (Hanover, N. H.).

Amerio, Luigi. Sui problemi di Cauchy e di Dirichlet per l'equazione di Laplace in due variabili. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 393–425 (1943).

The following summary is taken from a report on the paper, signed by Severi, Amaldi and Picone. The author considers the Dirichlet problem in the plane for regular domains with a single boundary and succeeds in giving a series for the solution by means of an iteration process of Neumann type, applied to a linear integral equation, which he deduces from the classical one (of the double layer) which does not lend itself, in general, to developments in series of the above type. Using his results, the author succeeds in establishing the validity of an algorithm deduced from the resolvent formula of the Cauchy problem for the same equation and for the same boundaries and in characterizing a class of the latter in correspondence with which the terms of the series are all calculated by means of rational operations from the Fourier series of the given function. *F. Zernike* (Baltimore, Md.).

Birindelli, C. Nuova trattazione di problemi al contorno di una striscia per l'equazione di Laplace in due variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 269–275 (1947).

Theorems on the existence and the uniqueness of the solutions of the equation $\Delta u = f(x, y)$ in an infinite strip, bounded by $y=0$ and $y=a$, satisfying given boundary conditions. For the fundamental problem these conditions are

$$(\alpha_0 u + \beta_0 \partial u/\partial y)_{y=0} = f_0(x), \quad (\alpha u + \beta \partial u/\partial y)_{y=a} = f(x),$$
$$\alpha_0 \neq 0,\ \beta_0 \neq 0,\ \alpha \neq 0,\ \beta \neq 0,\ \alpha_0^2 + \beta_0^2 = 1,\ \alpha^2 + \beta^2 = 1.$$

The problems of Dirichlet, Neumann and some mixed problems are particular or limiting cases. The author states conditions to be imposed on $f(x, y)$, much more general than those used in other treatments. Proofs are not given. *H. Bremekamp* (Delft).

Ghermanescu, Michel. Sur les valeurs moyennes des fonctions. Bull. Sci. École Polytech. Timişoara **12**, 17–48 (1945).

Différents types de relations linéaires entre les moyennes sphériques d'une fonction continue dans l'espace à p dimensions conduisent l'auteur à donner de plusieurs manières des propriétés caractéristiques des fonctions métaharmoniques, c'est-à-dire des solutions d'une équation (1) $\Delta_u^m + h_1\Delta_u^{m-1} + h_2\Delta_u^{m-2} + \cdots + h_m u = 0$, h_i constantes. La relation (2) $u(P) = \Phi_1(\rho)\mu_1(u, P) + \Phi_2(\rho)\mu_2(u, P) + \cdots + \Phi_m(\rho)\mu_m(u, P)$ est étudiée d'abord; $\mu_1(u, P)$ est la moyenne spatiale de u sur la sphère de centre P, de rayon R ($4\rho = R^2$), et les moyennes successives sont définies par

$$\mu_k(u, P) = \frac{p+2k-2}{R^{p+2k-2}} \int_0^R \mu_{k-1}(u, P) t^{p+2k-3} dt.$$

On suppose $\mu_1(u, P), \cdots, \mu_m(u, P)$ linéairement indépendants. Dans ces conditions (2) entraîne (1) et les $\Phi_k(\rho)$ sont nécessairement les solutions d'une certaine équation différentielle linéaire. De plus toute moyenne $\mu_k(u, P)$ s'exprime par une relation (3) $\mu_k(u, P) = \Phi_1^k(\rho)u_1(P) + \cdots + \Phi_m^k(\rho)u_m(P)$, où les $u_i(P)$ dépendent seuls de u, tandis que les $\Phi_i^k(\rho)$ sont solutions d'une équation différentielle formée simplement à partir des h_i. Réciproquement la relation (3) entraîne que $u(P)$, $u_i(P)$ soient des solutions d'une même équation (1). Même conclusion en partant de l'hypothèse que $\mu_k(u, P)$ est une fonction de ρ vérifiant une équation différentielle linéaire homogène d'ordre m indépendante de P, on encore qu'on a entre des moyennes successives une relation

$$(4)\quad u_0(P, \rho)\mu_k(u, P) + u_1(P, \rho)\mu_{k+1}(u, P) + \cdots + u_m(P, \rho)\mu_{k+m}(u, P) = 0.$$

La conclusion vaut encore lorsque'on part d'une équation fonctionnelle où n'intervient qu'une seule moyenne μ_k prise sur des sphères de centre P, de rayons ρ_i, $\rho_i = a_i\rho$, $4\rho = R^2$:

$$(5)\quad \psi_0^k(\rho)\mu_k(u, P, \rho) + \psi_1^k(\rho)\mu_k(u, P, \rho_1) + \cdots + \psi_m^k(\rho)\mu_k(u, P, \rho_m) = 0$$

qui fournit ainsi une autre propriété caractéristique des fonctions m métaharmoniques; les $\psi_i^k(\rho)$ sont nécessairement des fonctions particulières. Comme cas particulier de (5): pour que u soit m-harmonique, il faut et il suffit qu'il existe une combinaison linéaire homogène à coefficients constants des m moyennes $\mu_k(u, P, \rho_i)$ d'ordre k qui soit égale à u. Ce dernier résultat est à comparer avec une théorème plus général dû à G. Choquet et J. Deny [Bull. Soc. Math. France **72**, 118–140 (1944); ces Rev. **7**, 161] qui donne, au moins pour deux dimensions, les distributions les plus générales associées aux fonctions m-harmoniques. *P. Lelong* (Lille).

Bogdan, Gabriela. La dimostrazione di un teorema sopra le funzioni n-iperarmoniche. Disquisit. Math. Phys. **3**, 155–195 (1943).

Solutions of the differential equation $\Delta^n u = 0$, where Δ^n is the iterated Laplacian, are called n-hyperharmonic functions. In the present paper the author establishes existence theorems for this equation for general n and the case of two independent variables, when the values of u and of its first $n-1$ normal derivatives are prescribed on the

boundary of a circle. In previous work on this problem, the continuity properties required of the prescribed boundary values have been excessive, and the author relaxes these to the following extent: if $f_i(\theta)$, $i=0, 1, \cdots, n-1$, are periodic functions of θ in $0\leqq\theta\leqq 2\pi$, continuous together with their derivatives of orders through $n-k-1$, then there exists exactly one solution of $\Delta^n u=0$ in the circle $r\leqq R$, such that $\lim_{r\to R} \partial^i u(r,\theta)/\partial r^i = f_i(\theta)$, $i=0, 1, \cdots, n-1$. Examples are given in the case $n=3$ to show that in the absence of the stated continuity properties the solution may fail to exist. In proving this existence theorem, the author makes use of the method of Picone [Ann. Scuola Norm. Super. Pisa (2) 5, 213–288 (1936), in particular, pp. 257–262] for the solution of linear partial differential equations.

J. W. Green (Los Angeles, Calif.).

Differential Equations

***Kamke, E. Differentialgleichungen. Lösungsmethoden und Lösungen. Band I. Gewöhnliche Differentialgleichungen.** 3d ed. Mathematik und ihre Anwendungen in Physik und Technik. Band 18_1. Akademische Verlagsgesellschaft, Leipzig, 1944. xxvi+666 pp.

This remarkable reference book consists of two parts, of approximately equal lengths. As the author states, the purpose of the first part is to give, as far as possible, everything in the way of methods, theorems, etc., that can be useful to one who has a differential equation to solve, or who wishes to study the properties of the solutions of such an equation. The second part consists of a table of solutions of approximately 1600 particular differential equations.

How the purpose of the first part is achieved can be seen from the following brief summary of the contents of the first section [32 pages], devoted to equations of the first order. The section begins with a description of the geometrical significance of the equation $y'=f(x, y)$. Following this there are statements of the fundamental theorems concerning the existence and uniqueness of solutions, and references forward to theorems concerning the continuity of solutions with respect to initial values and other parameters. Here, as elsewhere in the book, there are no proofs, but there are abundant references to the literature. General methods of solution, by means of infinite series and successive approximations, are described quite fully. Paragraphs are devoted to the statement of theorems concerning the comparison and estimation of solutions, to the behavior of solutions for large values of the independent variable, and to singular solutions of equations of the form $f(x, y, y')=0$. The last seventeen pages of the section are devoted to a summary of the elementary methods for solving first order equations in finite form. The treatment throughout is careful and scholarly, and it is very well documented.

Subsequent sections deal in similar manners with systems of differential equations, systems of linear equations, the general equation of the nth order, the general linear equation of the nth order, differential equations of the second order, linear equations of the third and fourth orders, and numerical, graphical, and mechanical methods of solution.

What is here called the first part of the book closes with a discussion of eigenvalue problems, which runs to over a hundred pages. Here, in addition to the expected discussion of the classical boundary value problems for linear differential equations, we find discussions of many interesting eigenvalue problems of other kinds. For example, one paragraph is devoted to nonlinear eigenvalue problems of the second order.

The table which constitutes the second part of the book should be exceedingly useful. In a few cases a differential equation is listed, but dismissed with the statement that it cannot be solved in finite form in terms of known functions. For the most part, however, the information given concerning the solutions is surprisingly complete. Alternative forms of solution are given, singular solutions are noted, and in many cases the Green's functions for various boundary value problems are included. In the cases of the classical differential equations of mathematical physics the entries are especially ample. The entries under Bessel's equation run to about five pages, and those under Mathieu's equation to nearly four. Most of the entries in the table include references to the literature.

It seems to the reviewer that the author has been highly successful in achieving the end which he had in view. There can be no doubt that the book will be enormously useful, not only to engineers and physicists, but also to many professional mathematicians. The reviewer's only criticism is a tentative one. At present he has the impression that some parts of the table, such as the part devoted to equations of the first order, would have been easier to use if the differential equations had been ordered according to somewhat different principles. However, further experience with the use of the table may alter this impression.

L. A. MacColl (New York, N. Y.).

Goland, Martin, and Luke, Y. L. Note on the use of symmetric functions in the solution of linear differential equations with constant coefficients. J. Franklin Inst. 244, 221–223 (1947).

Brown, B. M. Solution of differential equations by operational methods. Math. Gaz. 31, 145–153 (1947).

Van den Dungen, F. H. Les équations canoniques du résonateur linéaire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 659–665 (1946).

Vallese, Lucio. Sulla analisi dei fenomeni di regime variabile nei sistemi lineari a costanti concentrate. Atti Accad. Sci. Napoli (3) 2, no. 3, 30 pp. (1946).

Fogagnolo, Bruna. Sulle vibrazioni dei sistemi a due gradi di libertà. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 5, 220–233 (1946).

The author considers the linear system with constant coefficients, F constant,

$$b_{11}q_1''+b_{12}q_2''+a_{11}q_1'+g_{12}q_2'+2c_{11}q_1=F\cos(vt-\alpha),$$
$$b_{22}q_2''+b_{12}q_1'+a_{22}q_2'+a_{22}q_1'+2c_{22}q_2=0,$$

with the conditions $a_{11}>0$, $a_{22}>0$, $a_{11}a_{22}-a_{12}^2>0$, $b_{11}>0$, $b_{22}>0$, $b_{11}b_{22}-b_{12}^2>0$, $c_{11}\geqq 0$, $c_{22}\geqq 0$, $F>0$, with regard to the influence of the coefficients on the boundedness of the solutions.

R. Bellman (Princeton, N. J.).

Roitenberg, J. N. Auto-oscillations of gyroscopic stabilizers. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 271–280 (1947). (Russian. English summary)

The equations of a gyroscopic ship stabilizer may be reduced to a system Σ of 5 linear equations of order one with constant coefficients in unknowns $\beta, x_1, \cdots, x_4$, where

β is an angular variable. Four of the equations are homogeneous and the fifth has a right-hand side of the form $K \operatorname{sgn} \beta$. Two solutions for $0 \leqq \beta \leqq \pi$, $-\pi \leqq \beta \leqq 0$, are pieced together to form a unique continuous solution. Periodic solutions exist and are determined and their stability discussed. A numerical example is computed to show that such stable solutions may exist. *S. Lefschetz.*

*Bol, G. **Invarianten linearer Differentialgleichungen.** Ber. Math.-Tagung Tübingen 1946, pp. 44–46 (1947).

By use of a specially chosen set of constants a_{ik}, the author replaces a general $(n+1)$th order linear ordinary differential equation by a system of $n+1$ first order differential equations in a particular form. After showing how the new system changes under change of independent variable, with corresponding change of dependent variable, he considers invariants under such changes, and how by means of them it can be determined whether the given equation can be transformed into given forms. The dependent variables of the system are introduced as coordinates in a projective space. A minimum of detail is given. *A. B. Brown.*

Berwald, L. Ueber Systeme von gewöhnlichen Differentialgleichungen zweiter Ordnung deren Integralkurven mit dem System der geraden Linien topologisch aequivalent sind. Ann. of Math. (2) **48**, 193–215 (1947).

Necessary and sufficient conditions are developed for the case that a system of $n-1 \geqq 2$ ordinary differential equations of order two,

$$d^2z^k/dx^2 = f^k(x, z, dz/dx), \quad k = 1, 2, \cdots, n-1,$$

where f^k stands for $f(x, z^1, z^2, \cdots, z^{n-1}, dz^1/dx, \cdots, dz^{n-1}/dx)$, can be transformed by coordinate transformations (x, z into $\bar{x}$, $\bar{z}$), into the system $d^2\bar{z}^k/d\bar{x}^2 = 0$. The character of these conditions can be seen from the special case when the equation $d^2y/dx^2 = f(x, y, dy/dx)$ is to be transformed into $d^2\bar{y}/d\bar{x}^2 = 0$. The conditions are

$$\partial^4 f/\partial y'^4 = f_{y'y'y'y'} = 0,$$

$$\left[\frac{d^2}{dx^2}f_{y'y'} - \left(\frac{d}{dx}f_{y'y'}\right)f_{y'}\right] - 4\left[\frac{d}{dx}f_{yy'} - f_{yy'}f_{y'}\right]$$

$$-3f_{y'y'}f_y + 6f_{yy} = 0.$$

These conditions associate a particular projective connection with the system of differential equations. *D. J. Struik.*

Satô, Tokui. Sur les points singuliers d'une équation aux différentielles totales. I. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2**, 139–150 (1942).

Satô, Tokui. Sur les points singuliers d'une équation aux différentielles totales. II. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **3**, 45–55 (1943).

Let $P(x, y, z)$ and $Q(x, y, z)$ be holomorphic functions in the neighborhood of $(0, 0, 0)$, with $P(0, 0, 0) = 0$, $P_z(0, 0, 0) = C$. Also let the total differential equation (1) $xdz = Pdx + xQdy$ be integrable. In the first paper the author shows (1) has a unique solution $z = \varphi(x, y)$ holomorphic in x, y with $\varphi(0, 0) = 0$ if the number C is not zero or a positive integer. If, furthermore, C is not a negative number, then (1) has a solution holomorphic in x, y, ζ of the form $z = \sum_{j=0}^{\infty} \varphi_j(x, y)\zeta^j$, where $\zeta = Kx^C$ and K is any arbitrary constant. The second paper is devoted to the case C a positive integer n. Assuming (1) cannot be reduced to an equation regular at $(0, 0, 0)$ and $C = n$, then (1) is shown to have a solution of the form $z = \sum_{j=1}^{\infty} \varphi_j(x, y)\zeta^j$, where $\zeta = Kx^n$. *F. G. Dressel.*

Hukuhara, Masuo. Sur les points singuliers d'une équation différentielle du premier ordre. V. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **3**, 67–73 (1944).

The equation $xy' = y\sum_{j,k,l} a_{jkl}x^jy^k(x^py^{-1})^l$ is studied from a point of view of securing formal solutions. Formal solutions are obtained in the following cases: $j + ka_{000} + l(\rho - a_{000}) \neq 0$; $a_{000} = 0$; $a_{000} = \rho$; a_{000} negative and rational; a_{000} rational and greater than ρ. *W. M. Whyburn* (Lubbock, Tex.).

Frère. Note sur l'intégrale de Cauchy d'une équation différentielle du premier ordre, de forme normale. Bull. Soc. Roy. Sci. Liége **14**, 419–423 (1945).

Zwirner, Giuseppe. A proposito di un teorema di unicità per gli integrali delle equazioni differenziali del primo ordine. Ann. Mat. Pura Appl. (4) **24**, 153–156 (1945).

The author proves a uniqueness theorem for the differential equation $y' = f(x, y)$, $0 \leqq x \leqq x_1$, $-\infty < y < \infty$, under conditions slightly more general than those of previous authors. *R. Bellman* (Princeton, N. J.).

Charpentier, Marie. Sur un faisceau de courbes intégrales de section non localement connexe. Bull. Sci. Math. (2) **70**, 151–155 (1946).

This paper gives an explicit example of a differential system of the type $dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$, where $f(x, y, z)$ and $g(x, y, z)$ are continuous in their three variables, such that a family of solutions passes through $(0, 0, 0)$ and intersects the plane $x = 1$ in the nonlocally connected continuum defined by $z = \sin \pi/y$, when $0 < |y| \leqq 1$; $-1 \leqq z \leqq 1$, when $y = 0$. *W. M. Whyburn.*

Magenes, E. Sopra un problema di T. Satô per l'equazione differenziale $y'' = f(x, y, y')$. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 258–261 (1947).

[For part I see the same vol., 130–136 (1947); these Rev. **8**, 515.] The author considers the differential equation $y'' = f(x, y, y')$ subject to the condition that a solution y passes through a given point and is tangent to a given curve. *R. Bellman* (Princeton, N. J.).

*Liapounoff, A. **Problème Général de la Stabilité du Mouvement.** Annals of Mathematics Studies, no. 17. Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1947. iv+272 pp. [paged 203–474]. $3.50.

This is a photographic reproduction of the paper of the same title [Ann. Fac. Sci. Univ. Toulouse (2) **9**, 203–474 (1907)]. The paper is a translation of a Russian memoir [Obščaya Zadača Ustoĭčivosti Dviženiya, Kharkow, 1892] and of an additional note [Comm. Soc. Math. Kharkow (2) **3**, 265–272 (1893)]. *N. Levinson* (Cambridge, Mass.).

Wintner, Aurel. A criterion for stable characteristic exponents. Quart. Appl. Math. **5**, 232–236 (1947).

Let $f(t)$ in the differential equation $x'' + f(t)x = 0$ have period 1. The characteristic roots s are considered. These are solutions of a quadratic equation $s^2 - 2\alpha s + 1 = 0$, where α can be determined, as was shown by Liapounoff, from a successive approximation procedure for obtaining two independent solutions of the differential equation [see, for example, Ince, Ordinary Differential Equations, London, 1927, p. 383]. Liapounoff used this method to show that, if $f(t)$ is positive and has average value less

than 4, the characteristic roots are of stable type. By modifying Liapounoff's proof the author obtains two different criteria which do not require that $f(t)$ is positive. Let μ be the average of $f(t)$ and let $M=\max|f(t)|$. Then if $\sum_2^\infty M^n/(2n-1)!<\mu\leqq 1$ or if $\sum_2^\infty M^n/(2n+1)!<4-\mu\leqq 2$, the roots are of stable type. *N. Levinson.*

Wintner, Aurel. Asymptotic integrations of the adiabatic oscillator. Amer. J. Math. **69**, 251–272 (1947).

Let $\phi(t)$ be real and of bounded variation over (t_0, ∞). Then corresponding to any solution $x(t)\not\equiv 0$ of

$$x''+(\omega^2+\phi(t))x=0,$$

where ω is a positive constant, there is a unique pair of constants, a and α, such that

$$x(t)=a\cos\left\{\alpha+\int_{t_0}^{t}(\omega^2+\phi(s))^{\frac{1}{2}}ds\right\}+\epsilon(t),$$

where $\epsilon(t)$ and its derivative tend to zero as $t\to\infty$. When $\phi(t)\epsilon L^2$, the integral in the above formula simplifies to $\omega t+(2\omega)^{-1}\int^t\phi(s)ds$. Other results are given. The generalization of Weyl's theorem on the Grenzkreis case given by the author has already been made and, in fact, for the L^p case where p is not restricted to be 2, by Bellman [Duke Math. J. **11**, 513–516 (1944); these Rev. **6**, 66]. *N. Levinson.*

Bellman, Richard. The boundedness of solutions of linear differential equations. Duke Math. J. **14**, 83–97 (1947).

Using the lemma that if y and z are positive functions and if $c>0$ then $y\leqq c+\int_{t_0}^t zy\,dt$ implies $y\leqq c\exp(\int_{t_0}^t z\,dt)$, the author proves a number of diversified results, some of which are known and some of which are improvements on known results. Simple proofs are given for theorems of Wintner, Dini-Hukuhara and the author. A result of Wiman is generalized giving the result that the equation $y''+(a^2+\phi(t)+\psi(t))y=0$ has bounded solutions if $a>0$, ϕ is of bounded variation and $|\phi|\leqq b^2<a^2$, and ψ is absolutely integrable. This result is extended to systems of second order equations. A result of Haupt is proved and a result of Poincaré on a linear system with coefficients tending to constant limits is sharpened. *N. Levinson.*

Cartwright, M. L., and Littlewood, J. E. On non-linear differential equations of the second order. II. The equation $\ddot{y}+kf(y)\dot{y}+g(y,k)=p(t)=p_1(t)+kp_2(t)$; $k>0$, $f(y)\geqq 1$. Ann. of Math. (2) **48**, 472–494 (1947).

The present paper, in contrast to part I [J. London Math. Soc. **20**, 180–189 (1945); these Rev. **8**, 68], takes the case $kf(y)$ always positive. It is assumed that $p_1(t)$ and $p_2(t)$ are periodic of least period λ and mean value zero. It is assumed that $g(y)$ has a restoring effect. In case $g(y)=y$ it has been shown by the reviewer [J. Math. Phys. Mass. Inst. Tech. **22**, 181–187 (1943); these Rev. **5**, 183] that there is a single periodic trajectory to which every trajectory converges as $t\to+\infty$. This case the authors call "convergence." They show that "convergence" occurs if $g(y)$ is nearly linear. "Convergence" also occurs under broad conditions if k is large. This latter result makes up most of the paper. The authors also give a particular example in which they rigorously establish non-"convergence" by showing the existence of 3 subharmonic periodic trajectories. This is done by use of a small parameter and depends on the nonvanishing of the relevant Jacobian. *N. Levinson* (Cambridge, Mass.).

Almeida Meneses, Pablo Rogerio. Subresonance for an equation of second order with forced and periodic vibrations, when the friction and the elastic force are not linear in the velocity. II. Revista Ci., Lima **49**, 87–166 (1947). (Spanish)

This is the second installment in the publication of a lengthy thesis on subresonance. [For the first part see the same vol., 71–80 (1947); these Rev. **8**, 583] The major part of this installment is devoted to a review of familiar matters relating to linear systems and to the phenomenon of subresonance. In the final pages the author begins the study of the equation $x''+\omega_0^2x+\mu f(x,x')=a\cos\omega t$, where $f(x,x')=(A+Bx+Cx^2)x'+\sum_{n=2}^N A_nx^n$. The publication of the thesis is to be continued. *L. A. MacColl.*

Rytov, S. M., and Zhabotinsky, M. E. Application of the small parameter method to systems close to those of Sturm-Liouville. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **11**, 135–140 (1947). (Russian. English summary)

The authors consider the nonlinear Sturm-Liouville system of the form (1) $L(y)-\rho y_u=f(y,\mu)$, where L is a self-adjoint partial differential operator, y is a vector with components y_1, y_2, y_3 and $f(y,\mu)$ is a vector function of y which reduces to zero for $\mu=0$. The boundary conditions are nonlinear, depend upon μ and have the property that for $\mu=0$ they become linear. The authors use the method of small parameters to obtain solutions of the nonlinear problem in terms of solutions of the linear problem, using standard perturbation technique. *R. Bellman* (Princeton, N. J.).

Stampacchia, G. Sulle condizioni che determinano gli integrali di un sistema di due equazioni differenziali ordinarie del primo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 411–418 (1947).

The initial theorem of this paper is as follows. If, on the strip $a\leqq x\leqq b$, $|y|<+\infty$, $|z|<+\infty$, the real-valued functions $f(x,y,z)$, $g(x,y,z)$ are continuous in (y,z) for fixed values of x, and measurable in x on $a\leqq x\leqq b$ for fixed (y,z), while there exists a nonnegative (Lebesgue) integrable function $\phi(x)$ such that

$$(1)\qquad |f(x,y,z)|\leqq\phi(x),\quad |g(x,y,z)|\leqq\phi(x),\quad a\leqq x\leqq b,$$

then on $a_1\leqq x\leqq a_2$, $a\leqq a_1<a_2\leqq b$, there exists at least one absolutely continuous solution $y=y(x)$, $z=z(x)$ of the differential system $y'=f(x,y,z)$, $z'=g(x,y,z)$ which satisfies given boundary conditions

$$(2)\qquad \alpha_iy(a_i)+\beta_iz(a_i)+\gamma_i=0,\quad \alpha_1\beta_2-\alpha_2\beta_1\neq 0,\quad i=1,2,$$

with real-valued coefficients α_i, β_i, γ_i. Generalizations of this theorem are established, in which the boundary conditions (2) are replaced by the requirement that the end-points $(a_i, y(a_i), z(a_i))$ lie on specified curves in the respective planes $x=a_i$ $(i=1,2)$, and in which (1) is replaced by the condition that $|f(x,y,z)|$, $|g(x,y,z)|$ are majorized by $\phi(x)\{|y|+|z|\}+\psi(x)$, where $\phi(x)$, $\psi(x)$ are nonnegative integrable functions on $a\leqq x\leqq b$. *W. T. Reid.*

Mikusiński, Jan G.-. Sur un problème d'interpolation pour les intégrales des équations différentielles linéaires. Ann. Soc. Polon. Math. **19**, 165–205 (1946).

The problem of interpolation here considered is the following one. Given a differential system

$$(1)\qquad dx_i/dt=\sum_{j=1}^n f_{i,j}(t)\cdot x_j+g_i(t),\qquad i=1,\cdots,n,$$

and n distinct values t_k, $k=1, \cdots, n$, does there exist a solution of (1) for which any predesignated subset of n of the quantities $x_i(t_k)$ take on prescribed values? In terms of a particular solution $\Psi_i(t)$ of (1) and n linearly independent solutions $\Phi_{i,j}(t)$, $j=1, \cdots, n$, of the "reduced" system, the general solution of (1) is expressible in the form $x_i(t)=\sum_{j=1}^{n}\lambda_j\Phi_{i,j}(t)+\Psi_i(t)$, the λ_j being parameters. Imposition of the boundary conditions leads readily to a "characteristic determinant" the vanishing of which precludes a unique solution. The elements of this determinant are in the set $\Phi_{i,j}(t_k)$. From the characteristic determinant the author derives certain sets of "déterminants combinés," and from analyses of these latter deduces a method for finding sufficient conditions for the non-evanescence of the characteristic determinant and hence for the unique solubility of the problem posed. *R. E. Langer* (Madison, Wis.).

Mikeladze, Sch. E. Solutions discontinues des équations différentielles linéaires ordinaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 789–792 (1947).

The ordinary linear differential equation of the nth order together with linear boundary conditions at two points is studied. The general solution of the homogeneous equation is expressed by a formula which admits possible discontinuities of the first kind in this solution and its first $n-1$ derivatives at points of the interval. With this formula, the solution of the homogeneous equation and boundary conditions is found by algebraic processes. The nonhomogeneous equation and boundary conditions is then handled by standard methods. The essential difference between this treatment and the ordinary Green's function treatment lies in the admission of discontinuities in the solution and its various derivatives rather than confining these discontinuities to a single derivative. This difference loses its significance when the single nth order equation is replaced by a system of equations of the first order.

W. M. Whyburn (Lubbock, Tex.).

Giuliano, Landolino. Sul teorema di confronto di Sturm. Boll. Un. Mat. Ital. (3) **2**, 16–19 (1947).

This note gives a new proof of the following theorem. Let the functions $p(x)$, $q(x)$, $q_1(x)$ be continuous in the interval (a, b). Let $y_1(x)$ be a solution, not identically zero, of $y''+p(x)y'+q(x)y=0$ and the boundary conditions $y_1(a)=y_1(b)=0$. If $q_1(x)\geqq q(x)$ in (a, b), and if $z_1(x)$ is a solution of $z''+p(x)z'+q_1(x)z=0$, then $z_1(x)$ has at least one zero in (a, b). *L. A. MacColl* (New York, N. Y.).

Pleijel, Åke. On the problem of the asymptotic distribution of proper values in the theory of ordinary differential equations. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **17**, no. 3, 16 pp. (1947).

The differential system for which the distribution of characteristic values is determined is based upon a differential equation (1) $F(u)+\lambda g(x)u=0$, in which $F(u)\equiv\sum_{r=0}^{n}\{f_r(x)u^{[r]}\}^{[r]}$, with $f_n(x)>0$, whereas $g(x)$ is permitted to change sign on the fundamental interval. The boundary conditions are taken to be of the Sturmian type and such that the system as a whole is self-adjoint. The operator $F(u)$ is furthermore taken to be positive definite.

The method, in outline, is the following. On each subinterval upon which $g(x)$ is of one sign, a function $\bar{g}(x)$ of the same sign, bounded from zero, but differing by arbitrarily little from $g(x)$ is chosen. Thus an auxiliary problem based upon the differential equation (2) $F(u)+\lambda\bar{g}(x)u=0$ is constructed, the boundary conditions of this consisting of the originally given ones plus the relations which impose continuity upon $u(x), u'(x), \cdots, u^{[2n-1]}(x)$ at the discontinuities of $\bar{g}(x)$. The characteristic values of this auxiliary problem are determinable by well-known methods. They are, however, also known to be defined by certain maximum-minimum relations. These latter are drawn upon for inequalities from which the characteristic values of (1) are inferable from those of (2). *R. E. Langer.*

Sansone, Giovanni. Sulle soluzioni di Emden dell'equazione di Fowler. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 163–176 (1940).

The author considers the equation

$$\frac{d}{dt}\left(t^{\alpha}\frac{dU}{dt}\right)+t^{\beta}U^{n}=0, \qquad a>0,\ n\geqq 0,$$

of importance in astrophysics, and proves that this equation has one and only one solution, defined in an interval $0<t<t_1$, with the property that $U>0$ in this interval and $\lim_{t\to 0} U=c$, $0<c<\infty$. The method used is that of successive approximations. [It might be noted that the substitution $t=1/s$ puts the equation into a form that makes the result more intuitive and simplifies the proof a bit.] It is then shown that the condition $2a-n+1>0$ is necessary and sufficient that $U(t)$ have a zero at a finite distance.

R. Bellman (Princeton, N. J.).

***Sansone, Giovanni. Sulle soluzioni di Emden dell'equazione di Fowler.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, p. 128. Edizioni Cremonense, Rome, 1942.

Announcement of results published in the paper reviewed above.

Sansone, G. Su un problema ai limiti per l'equazione differenziale $y^{(n)}(x)+\lambda(n-1)!\omega(x)y(x)=0$. Ann. Mat. Pura Appl. (4) **24**, 209–236 (1945).

By reducing the differential equation to a Fredholm integral equation and deriving the power series expression of $D(\lambda)$, the Fredholm determinant, the author shows that an application of some results of Laguerre concerning the zeroes of entire functions yields the existence of an infinite number of eigenvalues of the above differential equation, subjected to the boundary conditions $y(a)=y'(a)=\cdots=y^{(n-2)}(a)=0$, $y(b)=0$, $a\leqq x\leqq b$, $n\geqq 2$. He then discusses some comparison and oscillation theorems.

R. Bellman (Princeton, N. J.).

***Sansone, Giovanni. Problemi attuali sulla teoria delle equazioni differenziali ordinarie e su alcuni tipi di equazioni alle derivate parziali.** Atti Convegno Mat. Roma 1942, pp. 179–200 (1945).

Olevsky, M. N. Sur une généralisation d'un problème de Lamé-Darboux. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 685–688 (1947).

Let S_3 be a space of three dimensions with constant positive or negative curvature. The author has studied the problem of determining all triply orthogonal systems in S_3 for which the relation obtained by equating Beltrami's second differential operator to zero yields a complete separation of the variables. He states that the complete solution is contained in a list of thirty cases which he gives without proof. *F. W. Perkins* (Hanover, N. H.).

Kupradze, V. D. Certain new theorems on the equation of vibrations and their applications to boundary problems. Trav. Univ. Tbilissi **26A**, 1–11 (1945). (Russian. Georgian summary)

The author points out that if the solutions of Dirichlet's and Neumann's problems for the equation $\Delta u+k^2u=0$ are sought in the form of double and single layer potentials there result the following integral equations:

$$(D_i) \qquad \mu(p)-\int_S \mu(q)K(p,q)dS_q=F(p)$$

for Dirichlet's interior problem,

$$(D_e) \qquad \mu(p)+\int_S \mu(q)K(p,q)dS_q=F(p)$$

for Dirichlet's exterior problem,

$$(N_i) \qquad \nu(p)+\int_S \nu(q)K(q,p)dS_q=F(p)$$

for Neumann's interior problem,

$$(N_e) \qquad \nu(p)-\int_S \nu(q)K(q,p)dS_q=F(p)$$

for Neumann's exterior problem. Here S is a "surface of Liapounoff" [Privaloff and Sagatelian, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 429–436 (1941); these Rev. **3**, 125], which divides space into a bounded interior and an unbounded exterior region;

$$K(p,q)=(2\pi)^{-1}\frac{\partial}{\partial n_q}e^{-ikr(pq)}/r(pq);$$

r is the distance pq; $F(p)$ is a given function continuous on S; $\mu(p)$, $\nu(p)$ are unknown functions satisfying Hölder's condition.

If the homogeneous integral equations which correspond to the equations (D_i), (D_e), (N_i), (N_e) are denoted by (D_i^0). (D_e^0), (N_i^0), (N_e^0), the results of the present paper can be stated in the form of the two main theorems of the author. (A) The principal functions of the equation (D_e^0) are the limiting values of the potentials of single layers whose densities are linear combinations of the principal functions of the equation (N_i^0). (B) The principal functions of the equation (N_e^0) are the limiting values of the normal derivatives of the potentials of double layers whose densities are linear combinations of the principal functions of the equation (D_i^0).

H. P. Thielman (Ames, Iowa).

Mechvarišvili, Ya. G. On certain properties of regular solutions of the equation of vibrations. Trav. Univ. Tbilissi **26A**, 13–22 (1945). (Russian. Georgian summary)

The author considers the operation

$$L[\varphi(z)]=\varphi(z)+\overline{\varphi(z)}+\int_{z_0}^{z}H(\bar{z},z-\xi;\lambda/2)\varphi(\xi)d\xi+\int_{z_0}^{z}H(z,\bar{z}-\xi;\lambda/2)\overline{\varphi(\xi)}d\xi,$$

where $H(\bar{z},z-\xi;\lambda/2)=-(\partial/\partial\xi)J_0(\lambda[\bar{z}(z-\xi)]^{\frac{1}{2}})$ and J_0 is Bessel's function of order zero. The following theorem is proved. If $\varphi(z)$ is regular in a region S, the function $u(z,\bar{z})=L[\varphi(z)]$ is a regular real solution of the equation of vibration $\Delta u+\lambda^2u=0$ (λ real) for the same region S, and conversely, any regular real solution of the vibration equation in the region S can be represented as $u(z,\bar{z})=L[\varphi(z)]$. Here $\varphi(z)$ can be expressed in terms of $u(z,\bar{z})$ in the following form

$$\varphi(z)=f(z)+\int_{z_0}^{z}f(\xi)(\partial/\partial\xi)J_0(\lambda[\bar{z}_0(z-\xi)]^{\frac{1}{2}})d\xi,$$

where $f(z)=u(z,\bar{z}_0)-\frac{1}{2}u(z_0,\bar{z}_0)+i\beta$ and β is an arbitrary real constant.

Other results of the paper are extensions of one of the author's earlier works [Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 485–490 (1941); these Rev. **5**, 204]. *H. P. Thielman.*

***Magnus, Wilhelm. Über das Verhalten der Lösungen von $\Delta u+u=0$ im Unendlichen.** Ber. Math.-Tagung Tübingen 1946, pp. 103–104 (1947).

Let ω be a hemispherical region on the unit sphere $r=1$ of an n-dimensional space. In this space let u be a regular solution of the equation $\Delta u+u=0$ such that $|r^{(n-1)/2}u|\leqq M$ (a constant) and such that on all rays emanating from the origin and piercing the interior of ω we have $\lim_{r\to\infty} r^{(n-1)/2}u=0$. The author states without proof the conclusion $u\equiv 0$; he also gives some variations of this theorem.

F. W. Perkins (Hanover, N. H.).

Amerio, Luigi. Sull'integrazione dell'equazione $\Delta_2u-\lambda^2u=f$ in un dominio di connessione qualsiasi. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 79–102 (1945).

Basing himself on a paper by Picone [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 413–426 (1940); these Rev. **3**, 44] the author gives a procedure for solving the Dirichlet or Neumann problem in m variables by using the Laplace transformation, integrating over the given domain C. He considers a "multiply connected" C defined as follows: let the boundary T consist of $\mu+1$ hypersurfaces without multiple points, $T_0, T_1, \cdots, T_\mu$, of which $T_1, \cdots, T_\mu$ are inside T_0 and outside each other. For $m>2$ the property thus defined has nothing to do with the usual multiplicity of connection which is of importance in solving the Neumann problem. *F. Zernike* (Baltimore, Md.).

Amerio, Luigi. Sul calcolo delle soluzioni dei problemi al contorno per le equazioni lineari del secondo ordine di tipo ellittico. Amer. J. Math. **69**, 447–489 (1947).

In a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 175–182 (1946); these Rev. **8**, 383] the author gave an account of a generalisation of Picone's method of solving the problem of Dirichlet for a linear partial differential equation of elliptic type. Only a brief account was given in that paper and fuller details were promised later. The paper under review fulfils that promise.

E. T. Copson (Dundee).

Amerio, Luigi. Sull'equazione di propagazione del calore. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 84–120 (1946).

Results stated in two earlier papers [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 346–352, 544–548 (1946); these Rev. **8**, 274] are proved in the present paper.

F. G. Dressel (Durham, N. C.).

Karimov, Dsh. H. Sur les solutions périodiques des équations différentielles non linéaires du type parabolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 119–121 (1947).

In an earlier paper [same C. R. (N.S.) **28**, 403–406 (1940); these Rev. **2**, 204] the author proved that for u sufficiently

small the following boundary value problem has a unique solution:

$$\frac{\partial}{\partial x}\left\{P(x)\frac{\partial Z}{\partial x}\right\}-\frac{\partial Z}{\partial t}=\Phi(x,t)+uf(Z),$$

$$Z(0,t)=Z(\pi,t)=0,\quad Z(x,0)=Z(x,1).$$

The present paper removes the restriction that u is sufficiently small. *F. G. Dressel* (Durham, N. C.).

Brinkley, Stuart R., Jr. Heat transfer between a fluid and a porous solid generating heat. J. Appl. Phys. **18**, 582–585 (1947).

Anzelius [Z. Angew. Math. Mech. **6**, 291–293 (1926)] has developed a theory for the transfer of heat between a porous solid and a fluid flowing through it. These results are extended to the case where the solid generates heat, the heat source function being taken as a linear function of the temperature of the solid. The heat transfer equations are simplified by a suitable transformation and the resulting hyperbolic partial differential equations solved by the Riemann integration method. The expressions for the temperatures of the solid and the fluid are given in closed form. *E. N. Nilson* (Hartford, Conn.).

Kohn, W. A note on Weinstein's variational method. Physical Rev. (2) **71**, 902–904 (1947).

Results of D. H. Weinstein [Proc. Nat. Acad. Sci. U. S. A. **20**, 529–532 (1934)] are used to prove the following theorems. Let E_n be the nth energy level of a Schroedinger equation $H\psi=E\psi$. If a lower bound L_{n+1} is known for the $(n+1)$th level and φ is any normalized trial-function satisfying $\bar{H}=\int\varphi^* H\varphi dq\leqq L_{n+1}$, then

$$E_n\geqq\bar{H}-\{\overline{H^2}-(\bar{H})^2\}/(L_{n+1}-\bar{H}).$$

Analogously, if U_{n-1} is an upper bound for E_{n-1}, then

$$E_n\leqq\bar{H}+\{\overline{H^2}-(\bar{H})^2\}/(\bar{H}-U_{n-1}),$$

provided $\bar{H}\geqq U_{n-1}$. *L. Hulthén* (Lund).

Cohn, Harvey. The Riemann function for $\partial^2u/\partial x\partial y+H(x+y)u=0$. Duke Math. J. **14**, 297–304 (1947).

The principal result established in this paper is that for hyperbolic equations of the type $\partial^2u/\partial x\partial y+H(x+y)u=0$ an explicit formula for the Riemann function $R(x,y;x',y')$ (which takes the value 1 when $x=x'$ or $y=y'$) can be given when H is of the form $H(r)=-\lambda(\lambda+1)\mu^2/\sinh^2\mu(r+\nu)$, where λ, μ and ν are constants. The formula is

$$R(x,y;x',y')=F(1+\lambda,-\lambda;1;V),$$

where

$$V=-\frac{\sinh\mu(x-x')\sinh\mu(y-y')}{\sinh\mu(x+y+\nu)\sinh\mu(x'+y'+\nu)},$$

and F is the hypergeometric function with the arguments specified. The paper also includes a discussion of a Fourier integral expression for Riemann's function with an arbitrary $H(r)$. [Cf. B. Riemann, Gesammelte Mathematische Werke, Leipzig, 1892, pp. 156–181.] *S. Chandrasekhar.*

Friedlander, F. G. On the integrals of a partial differential equation. Proc. Cambridge Philos. Soc. **43**, 348–359 (1947).

The equations considered are of the form

(E) $$u_{tt}=u_{xx}+g(x)u.$$

In the case $g\equiv0$ we have the ordinary wave equation, every solution of which is the sum of two progressive wave solutions, which satisfy, respectively, the first order equations (1) $u_t=\pm u_x$. The author shows that similarly a solution of the general equation (E) can be represented as the sum of two "progressive waves." Here progressive waves are defined as those special solutions which are constant along some characteristic line $x\pm t=\text{constant}=c$. The progressive wave solutions in the general case satisfy instead of (1) the integrodifferential equations

$$\partial u(x,t)/\partial t=\pm\partial u(x,t)/\partial x+\int_{x\mp t}^{x}r_u(x,y,0)u(y,t)dy,$$

where $r(x,y,s-t)$ is the Riemann function belonging to (E). *F. John* (New York, N. Y.).

Friedlander, F. G. Simple progressive solutions of the wave equation. Proc. Cambridge Philos. Soc. **43**, 360–373 (1947).

The author asks for functions $u(x,y,z,t)$ of the form $u=u_1(x,y,z)F(t-f(x,y,z))$, which for an arbitrary function F are solutions of the wave equation $u_{tt}=u_{xx}+u_{yy}+u_{zz}$. He finds by the methods of differential geometry that for such solutions the wave fronts $f(x,y,z)=t$ are necessarily the fourth (or lower) order algebraic surfaces known as cyclides of Dupin. These include as special or degenerate cases wave fronts of the form of a plane, sphere, cone and torus. The corresponding functions u_1 are determined as well. *F. John* (New York, N. Y.).

Difference Equations, Special Functional Equations

Cârstoiu, Ion. Sur quelques équations fonctionnelles et le calcul symbolique. C. R. Acad. Sci. Paris **224**, 1199–1200 (1947).

Five functional equations are solved by means of the Laplace transform. The first three equations are the classical ones: $f(x+y)=f(x)+f(y)$, $f(x+y)=f(x)f(y)$ and $f(x+y)+f(x-y)=2f(x)f(y)$. It is assumed that the derivative $f'(x)$ exists; this alone yields the solutions in a trivial manner. The use of the Laplace transform is thus redundant. The remaining two equations $f(x+1)-f(x)=x^n$ and $a_0f(x+n)+\cdots+a_nf(x)=f_1(x)$ are solved formally in a familiar manner. *M. Kac* (Ithaca, N. Y.).

Alaci, V. La solution analytique d'un système fonctionnel. Bull. Sci. École Polytech. Timişoara **11**, 174–178 (1944).

By comparing coefficients in power series it is shown that the general analytic solution of the functional equations

$$\varphi(x+y)=\varphi(x)\varphi(y)-\psi(x)\psi(y),$$
$$\psi(x+y)=\varphi(x)\psi(y)+\varphi(y)\psi(x)$$

is of the form $\varphi(x)=e^{ax}\cos bx$, $\psi(x)=e^{ax}\sin bx$. [For the same result under less restrictive conditions see Anghelutza, Mathematica, Timişoara **19**, 19–22 (1943); these Rev. **5**, 72]. *F. John* (New York, N. Y.).

Ghermanescu, Michel. Sur une équation fonctionnelle. Bull. Sci. École Polytech. Timişoara **10**, 258–272 (1941).

The author finds the general solution $f(n)$ of the functional equation

(1) $$f(n+l)f(n-l)-f^2(n)=A(n)B^{n-1}(n)$$

(l a fixed constant, $A(n)$ and $B(n)$ given functions of period l). The method of solution consists in reduction of (1) to a linear equation. The same method can be applied

to all equations of the type

$$\text{determinant}_{i,k=0,\cdots,p} f(n+(p-i-k)l)=A(n)B^{n-1}(n).$$

Most of the paper is devoted to a detailed discussion of sequences of polynomials $P_n(x)$ satisfying a recursion formula of the type

$$P_{n+1}(x)P_{n-1}(x)-P_n^2(x)=A(x)B^{n-1}(x) \tag{2}$$

with given polynomials $A(x)$ and $B(x)$. Formula (2) may be considered as a special case of (1). Conditions are derived under which a sequence $P_0, P_1, P_2, \cdots$ satisfying (2) can be continued to the left, i.e., polynomials $P_{-m}, P_{-m+1}, \cdots, P_{-1}$ satisfying (2) can be found. *F. John* (New York, N. Y.).

Ghermanescu, Michel. Sur une équation fonctionelle. Bull. Sci. École Polytech. Timişoara **11**, 30–34 (1943).

The general continuous solution $\Phi(x, y, z, t)$ of the system of equations

$$\Phi(x, y, z, t)=\Phi(x, t, z, y)-\Phi(y, t, z, x),$$
$$\Phi(x, y, z, t)=\Phi(y, x, z, t)=-\Phi(x, y, t, z)$$

is found to be representable in the form

$$\Phi(x, y, z, t)=A(x, y, z, t)-A(y, z, t, x)+A(z, t, x, y)$$
$$-A(t, x, y, z)+A(x, y, t, z)-A(y, t, z, x)$$
$$+A(t, z, x, y)-A(z, x, y, t)$$

with an arbitrary continuous function $A(x, y, z, t)$, which is symmetric in y and t. *F. John* (New York, N. Y.).

Ghermanescu, Michel. Sur quelques équations fonctionnelles. Bull. Sci. École Polytech. Timişoara **11**, 181–184 (1944).

The author obtains the "general" solution of functional equations of the type

$$m^r f(m^{\alpha_1}x_1, m^{\alpha_2}x_2, \cdots, m^{\alpha_n}x_n)=f(x_1, \cdots, x_n)+F(x_1, \cdots, x_n)$$

with a given function F and given positive constants $m, r, \alpha_1, \cdots, \alpha_n$. The validity of the author's conclusions seems to depend on assumptions about the behaviour of f at the origin, which are not stated explicitly.

F. John (New York, N. Y.).

Ascoli, Guido. Sopra un'equazione funzionale. Ann. Mat. Pura Appl. (4) **24**, 195–208 (1945).

This is almost the same as the author's paper in Portugaliae Math. **4**, 145–157 (1945); these Rev. **6**, 271.

R. P. Boas, Jr. (Providence, R. I.).

Christov, Chr. Sur une équation fonctionnelle de M. L. Tchakalov. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 45–53 (1946). (Bulgarian. French summary)

M. L. Tchakalov a proposé le problème suivant: trouver la forme la plus générale de la fonction $f(x)$ analytique dans l'intervalle $0<x<\pi$, satisfaisante à l'équation fonctionnelle $f(\alpha)+f(\gamma)=2f(\beta)$, α, β, γ étant les angles d'un triangle, dont les côtés forment une série arithmétique. Nous avons démontré que cette fonction est donnée par l'expression

$$f(x)=c+\sum_{n=1,2,\cdots}^{\infty} c_n T(v^n)=c\sum_{k=0,1,\cdots}^{\infty} S(v^{2^k}),$$

$$v=(3^{-\frac{1}{2}}\cot\tfrac{1}{2}x-1)/(3^{-\frac{1}{2}}\cot\tfrac{1}{2}x+1)$$
$$=\sin(\tfrac{1}{6}\pi-\tfrac{1}{2}x)/\sin(\tfrac{1}{6}\pi+\tfrac{1}{2}x),$$

où c est une constante arbitraire; $T(v)$ la fonction spéciale $T(v)=v+v^3+v^4+\cdots+v^{2^k}+\cdots$, $|v|<1$; $S(v)$ une fonction impaire soumise à la condition unique d'être analytique dans l'intervalle $-1<v<1$; et $c_1, c_2, c_3, \cdots$ les coefficients de son développement en série de Taylor, $S(v)=c_1v+c_3v^3+c_5v^5+\cdots$, convergente dans le voisinage de l'origine $v=0$.

Author's summary.

Robinson, L. B. A functional equation with negative exponent. Revista Ci., Lima **48**, 101–107 (1946).

The method of successive approximations is applied to the adjoint functional equations

$$u'(x)=\frac{\lambda}{1+x^2}\cdot\frac{P_n(x)}{Q_n(x)}\cdot u(x^{-2}), \qquad u'(x^{-1})=\frac{-\lambda}{1+x^2}\cdot\frac{P_n(x^{-1})}{Q_n(x^{-1})}\cdot u(x^2).$$

Assumptions: (i) P_n, Q_n are polynomials of degree n; (ii) $Q_n(0)\neq 0$; (iii) setting $P_n(x^{-1})/Q_n(x^{-1})=\bar{P}_n(x)/\bar{Q}_n(x)$, then $\bar{Q}_n(0)\neq 0$; (iv) x is restricted to the star-domain D determined by the points that are the roots of the equations $(1+x^{2r})Q_n(x^{2r-1})=0$, $(1+x^{2r})Q_n(x^{-2r-1})=0$. Solutions $u(x)$, $u(x^{-1})$ are obtained as power series in λ with functions of x as coefficients, and are shown to be joined on the unit circle.

I. M. Sheffer (State College, Pa.).

Integral Equations

Zaanen, A. C. On the theory of linear integral equations. VIII. Nederl. Akad. Wetensch., Proc. **50**, 465–473 = Indagationes Math. **9**, 271–279 (1947).

[Paper VII of this series appeared in the same Proc. **50**, 357–368 = Indagationes Math. **9**, 215–226 (1947); these Rev. **8**, 586.] The author extends some of the results of paper III of the series [same Proc. **49**, 292–301 = Indagationes Math. **8**, 161–170 (1946); these Rev. **8**, 28] to systems of linear integral equations. No new questions of principle are involved. *F. Smithies* (Cambridge, England).

Zaanen, A. C. On the theory of linear integral equations. VIIIa. Nederl. Akad. Wetensch., Proc. **50**, 612–617 = Indagationes Math. **9**, 320–325 (1947).

[Cf. the preceding review.] The author continues the extension of the results of paper III of the series to systems of linear integral equations. He shows that his results are in many respects an improvement on those obtained by J. E. Wilkins [Duke Math. J. **11**, 155–166 (1944); these Rev. **5**, 267]. *F. Smithies* (Cambridge, England).

***Bückner, Hans. Lösung von Integralgleichungen.** Ber. Math.-Tagung Tübingen 1946, pp. 56–58 (1947).

In connection with the integral equation

$$y(s)=f(s)+\lambda\int_a^b K(s, t)y(t)dt,$$

the author considers the iteration formula

$$u_{n+1}(s)=\theta u_n(s)+(1-\theta)\lambda\int_a^b K(s, t)u_n(t)dt+(1-\theta)f(s)$$

instead of the usual Liouville formula to which it reduces when $\theta=0$; and states a necessary condition for the convergence of $u_0, u_1, u_2 \cdots$ for an arbitrary u_0 in the form $|\theta+\lambda_k^{-1}(1-\theta)\lambda|<1$ for all the eigenvalues λ_k of the kernel $k(s, t)$. The paper also contains a description of an algebraic method of estimating these eigenvalues. *H. R. Pitt.*

Conte, Luigi. Su una particolare equazione integrale in n variabili. Boll. Un. Mat. Ital. (3) **2**, 25–28 (1947).

The equation under discussion is the Volterra equation

$$\varphi(x, y)=f(x, y)+\int_0^x dx_1\int_0^y K(x, y, x_1, y_1)\varphi(x_1, y_1)dy_1,$$

where the kernel has the form $e^{m_1\varphi_1(x)+m_2\varphi_2(y)+m_3\varphi_3(x_1)+m_4\varphi_4(y_1)}$, m_i constants and φ_i given functions. The resolvent kernel is readily found and involves the Bessel function of order zero. This is a generalization of a result due to Lodi [Boll. Un. Mat. Ital. (2) **3**, 391–393 (1941); these Rev. **3**, 151].

I. A. Barnett (Cincinnati, Ohio).

Parodi, Maurice. Application du calcul symbolique à la recherche de solutions d'équations intégrales. Revue Sci. **85**, 41 (1947).

The two integral equations

$$\int_0^t J_0[2\{x(t-x)\}^{\frac{1}{2}}]f(x)dx=\sum_{n=0}^{k} t^n/\Gamma(n+1),$$

$$f(t)-\int_0^t f(\{t^2-x^2\})^{\frac{1}{2}}J_1(x)dx=t^{2n},$$

where n is a positive integer, are solved by means of operational calculus. The method demonstrates the use of two of the less-known properties of Laplace transforms.

R. V. Churchill (Ann Arbor, Mich.).

Parodi, Maurice. Sur des équations intégrales singulières ayant des solutions communes. C. R. Acad. Sci. Paris **225**, 35–36 (1947).

The author inquires when $F(y)=\lambda\int_0^\infty K(y, z)F(z)dz$ has a solution independent of λ. He fails to answer the question except in the case when $K(y, z)\supset y^\lambda e^{-zy^\lambda}$, in which case he claims $F(y)=\int_0^\infty\{y^s/\Gamma(1+s)\}ds$. This is correct formally, but the conditions of validity are not given.

H. Pollard (Ithaca, N. Y.).

Eriksson, H. Adolf S. On the stationary energy and spatial distribution of neutrons in a medium of infinite size. Ark. Mat. Astr. Fys. **34B**, no. 2, 8 pp. (1947).

I. Waller [same Ark. **34A**, nos. 3, 4, 5 (1947); these Rev. **8**, 587] has shown that the solution of the problem mentioned in the title depends on solving a certain integral equation. The present author shows that an approximate solution of this integral equation can be deduced if one can solve the equations

$$g(x, r)=(1-\alpha^2)^{-1}\int_{x-a}^{x} dx'\int_0^\infty dr'g(x', r')K(r, r'), \qquad x>a,$$

$$g(x, r)=(1-\alpha^2)^{-1}\Big\{\kappa\int_0^\infty (e^{-r'}/r')K(r, r')dr' + \int_0^x dx'\int_0^\infty dr'g(x', r')K(r, r')\Big\}, \quad 0<x<a,$$

where $\alpha^2=e^{-\kappa}$ and where the nucleus is defined by

$$K(r, r')=\tfrac{1}{2}\int_{|r-r'|}^{r+r'} t^{-1}e^{-t}dt.$$

By using the Fourier sine-transform

$$f(x, s)=(2/\pi)\int_0^\infty g(x, r)\sin rs\,dr,$$

it follows that

$$f(x, s)=(1-\alpha^2)^{-1}s^{-1}\tan^{-1}s\int_{x-a}^{x} f(x', s)dx', \qquad x>a,$$

$$f(x, s)=(1-\alpha^2)^{-1}\Big\{(2k/\pi s)(\tan^{-1}s)^2 + s^{-1}\tan^{-1}s\int_0^x f(x', s)dx'\Big\}, \quad 0<x<a.$$

This system can be solved by the method given by the author in an earlier paper [same Ark. **33B**, no. 5 (1946); these Rev. **8**, 212].

E. T. Copson (Dundee).

Pitt, H. R. On a class of linear integro-differential equations. Proc. Cambridge Philos. Soc. **43**, 153–163 (1947).

This paper deals with the nonhomogeneous integro-differential equation

$$\sum_{r=0}^{R}\int_{-\infty}^{\infty} f^{(r)}(x-y)dk_r(y)=g(x), \tag{1}$$

and forms the sequel to the author's paper [same Proc. **40**, 199–211 (1944); these Rev. **6**, 273] in which the corresponding homogeneous equation was studied. The author now gives existence theorems for solutions of (1) with hypotheses and conditions on the solutions similar to those in the earlier paper. He also gives theorems dealing with the semi-infinite interval in which $k_r(y)$ is constant for $y<0$ and considers solutions that satisfy (1) for all (or almost all) x and also solutions that satisfy (1) for $x>0$. He finally uses these results to prove the following closure theorem. Suppose that the kernels $k_r(y)$ $(r=1, \cdots, R)$ are of bounded variation on $0\leqq y\leqq A$ and that $k_R(y)$ is discontinuous at 0 and A; let

$$K(\omega)=\sum_{r=0}^{R}\omega^r\int_{0^-}^{A^+} e^{-\omega y}dk_r(y),$$

and let the set of zeros of the entire function $K(\omega)$ be $\omega_1, \omega_2, \cdots$ with respective multiplicities $\gamma_1, \gamma_2, \cdots$. Then the set of functions $e^{\omega_n x}x^j$, $0\leqq j<\gamma_n$, $n=1, 2, \cdots$ is closed and independent in L_p over any interval of length A for any $p\geqq 1$ (j is an integer).

R. H. Cameron.

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XV. Astrophys. J. **105**, 424–434 (1947).

The discussion given in papers XI and XIV of this series [same J. **104**, 101–132 (1946); **105**, 164–203 (1947); these Rev. **8**, 59, 467] is extended to the case of an atmosphere scattering radiation according to Rayleigh's law, partial elliptic polarization of the radiation field being allowed for. Three of the equations of transfer are found to be of the same form as in the case of partial plane polarization [paper XI] and the fourth, which is new and independent of the others, can be solved by the same technique [paper XIV].

F. Smithies (Cambridge, England).

Chandrasekhar, S., and Breen, Frances H. On the radiative equilibrium of a stellar atmosphere. XVI. Astrophys. J. **105**, 435–440 (1947).

[Cf. the preceding review.] The function $H(\mu)$ introduced in paper XIV of this series and satisfying the functional equation

$$H(\mu)=1+\mu H(\mu)\int_0^1 \frac{H(\mu')\Psi(\mu')}{\mu+\mu'}d\mu'$$

is tabulated for characteristic functions $\Psi(\mu)$ corresponding to transfer problems involving Rayleigh's phase function and Rayleigh scattering, the polarization of the scattered radiation being allowed for. The corresponding laws of darkening are also tabulated, together with the degree of polarization of the emergent radiation. *F. Smithies.*

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XVII. Astrophys. J. 105, 441–460 (1947).

[Cf. the two preceding reviews.] In the solutions of the equations of transfer found in previous papers of this series [see, e.g., paper XIV cited above] relationships were found to exist in the case of constant net flux between the law of darkening and the law of diffuse reflection. For instance, if the scattering is isotropic, the law of darkening $I(0, \mu)$ is connected with the scattering function $S(\mu, \mu_0)$ by the relations $I(0, \mu)=\frac{1}{4}\sqrt{3}FH(\mu)$, $(\mu^{-1}+\mu_0^{-1})S(\mu, \mu_0)=H(\mu)H(\mu_0)$. where $H(\mu)$ satisfies the functional equation

$$H(\mu)=1+\tfrac{1}{2}\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}d\mu'.$$

In the present paper a relation is proved to exist in the general case between $I(0, \mu)$ and the azimuth-independent term

$$S^{(0)}(\mu, \mu_0)=(2\pi)^{-1}\int_0^{2\pi} S(\mu, \varphi; \mu_0, \varphi_0)d\varphi$$

of the scattering function $S(\mu, \varphi; \mu_0, \varphi_0)$. This relation, whose explicit form is too complex to give here, is proved directly by using the fact that the emergent radiation from a semi-infinite atmosphere is unaltered by the addition or removal of a layer of arbitrary optical thickness. The result is further generalized to cases when polarization is taken into account, so that a scattering matrix has to be used instead of a scattering function.

The author then derives by similar considerations functional equations connecting the diffuse reflection of radiation from a plane-parallel atmosphere of finite optical thickness with the transmission of radiation through such an atmosphere. He also obtains functional equations governing the radiation field in the interior of such an atmosphere.

F. Smithies (Cambridge, England).

Chandrasekhar, S., and Breen, Frances H. On the radiative equilibrium of a stellar atmosphere. XVIII. Astrophys. J. 105, 461–470 (1947).

[Cf. the three preceding reviews.] In the present paper are tabulated the functions $\mathfrak{J}(\alpha, \beta)$ and $\mathfrak{F}(\alpha, \beta)$ introduced in paper VII of this series [same J. 101, 328–347 (1945); these Rev. 6, 244]. Integrals over the interval $(0, \infty)$ are evaluated by using a quadrature formula based on the zeros of the Laguerre polynomials. *F. Smithies.*

Functional Analysis

Mikusiński, Jan G.-. Les méthodes algébriques dans l'analyse fonctionnelle. C. R. Acad. Sci. Paris 224, 1685–1687 (1947).

The author opens with a natural definition of a ring of $n$tuples $(a_1, \cdots, a_n)$, where $a_i \varepsilon R_i$, R_i a ring. Later he requires a distinguished subset A_i (of R_i) which is composed of nonnegative transitively ordered elements with the order invariant under translations and left or right multiplication by elements of A_i. Furthermore, λ real and positive implies $\lambda x \varepsilon A_i$ for all $x \varepsilon A_i$. Each $x \varepsilon R_i$ has an associated $|x| \varepsilon A_i$ where $|x|$ satisfies the formal relations ascribed to a norm or absolute value. The author introduces a topology into R_i which the reader will recognize is the sequential relatively uniform topology due to Moore where the dominants εy are in A_i. In the application to the Heaviside calculus, R_1 is the complex number ring and R_2 that of complex-valued functions with multiplication defined as convolution. [The author does not use a conjugate in his definition of convolution and asserts that x may be any complex-valued function of a real variable. In corrected form this type of ring for the Heaviside calculus has been recognized so that any novelty lies in the suggestion of the relatively uniform topology.] *D. G. Bourgin* (Urbana, Ill.).

Ganapathy Iyer, V. The strongest topology on a linear space subject to given condition. Proc. Indian Acad. Sci., Sect. A. 25, 529–538 (1947).

The author considers the partially ordered set of all T_1 topologies definable on a fixed linear space E. His principal result is the following. Let F be a countable family of linear functionals defined on E. Let $T_0(E, F)$ be that topology having the fewest open sets among all topologies in E with respect to which the members of F are continuous. Then $T_0(E, F)$ is the lattice product of a certain noncountable subfamily of a certain family of norm topologies in E but is not the lattice product of any countable subfamily of this family. *G. W. Mackey* (Cambridge, Mass.).

Pinsker, A. G. Completely linear functionals in K-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 299–302 (1947).

Let X be a partially ordered linear space. The author defines convergence for transfinite sequences of elements of X by generalizing the lim sup = lim inf criterion for convergence of ordinary sequences of real numbers. He then defines an additive functional f on X to be completely linear if $f(\lim x_\xi)=\lim f(x_\xi)$ for every convergent transfinite sequence x_ξ. A modification of this definition in which infinite values are allowed and $f(x)$ is not required to have a value for all x leads to the notion of generalized completely linear functional. The author gives an example of a linear functional which is not completely linear and states without proof a number of theorems dealing with the notions just described. These involve the existence of nontrivial completely linear and generalized completely linear functionals, the representation of functionals as sums of functionals of special kinds, etc. *G. W. Mackey* (Cambridge, Mass.).

Vulich, B. Sur quelques opérations non-linéaires dans les espaces semi-ordonnés linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 475–478 (1946).

This paper concerns the analytical representation of a certain type of nonlinear operator u with domain and range in partially ordered linear vector spaces X, Y with a unit (the spaces X, Y satisfy the axioms I–V of Kantorovitch). The function u is partially additive if $u(x_1+x_2)=u(x_1)+u(x_2)$ whenever x_1 and x_2 are disjoint; it is disjoint if $u(x_1)$ and $u(x_2)$ are disjoint whenever x_1 and x_2 are. The function u is uniformly continuous if, for every constant C, $\lim_{\delta \to 0} |u(x_1)-u(x_2)|$ is zero in the (o)-topology where $|x_1-x_2|<\delta \cdot 1$ and $|x_1|$, $|x_2|<C \cdot 1$. A partially additive disjoint uniformly continuous operator is called an operator (ADC). If in addition the (o)-convergence of x_n to x implies

that of $u(x_n)$ to $u(x)$ then u is an operation $(ADCO)$. A function φ which maps quasi-units in X into quasi-units in Y is called unitary if $\varphi(e_1+e_2)=\varphi(e_1)+\varphi(e_2)$ for e_1, e_2 disjoint and if the (o)-convergence of e_n to e implies the (o)-convergence of $\varphi(e_n)$ to $\varphi(e)$ it is called a measure function. In terms of these notions it is stated without proof that the general analytical expression for an operation (ADC) $(ADCO$, respectively) is given by the Radon integral $u(x)=\int_{-\infty}^{\infty}Y(\lambda)d\varphi(e_\lambda)x$, where $Y(\lambda)$ is uniformly continuous on each finite segment whose values lie in Y, φ is a unitary function (a measure function, respectively) and $Y(0)=0$. If in addition $Y(\lambda)=\int_0^\lambda Z(t)dt$, where $Z(t)$ is uniformly continuous on each finite segment, then for each x in X the Fréchet differential $d_xu(h)$ of u exists. *N. Dunford.*

Eberlein, W. F. Weak compactness in Banach spaces. I. Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947).

L'auteur démontre un résultat inattendu: pour qu'une partie M d'un espace de Banach soit (bi)compacte pour la topologie faible, il suffit qu'elle soit compacte au sens de Fréchet, c'est-à-dire que toute suite de points de M admette une valeur d'adhérence (faible). La démonstration fait appel aux théorèmes les plus profonds de Banach sur la topologie faible. [~~Le nom de "théorème de Tychonoff-Alaoglu," donné par l'auteur au théorème sur la compacité faible de la sphère unité dans le dual d'un espace de Banach, n'est guère justifié, car le théorème a été énoncé avant Alaoglu par N. Bourbaki, C. R. Acad. Sci. Paris 206, 1701–1704 (1938).~~] *J. Dieudonné* (Nancy).

Dunford, Nelson. Direct decompositions of Banach spaces. Bol. Soc. Mat. Mexicana **3**, 1–12 (1946).

Let X be a Banach space and G a family of linear operators on X to X. Write $M(G)=\{x\mid T(x)=\theta, T\in G\}$ and $N(G)$ for the closed linear extension of the union of the ranges of $\{T\mid G\}$. Problem (A): when is $X=M(G)\oplus N(G)$? Problem (B) is the special case $G\equiv T$. Other formulations of (B) for N-dimensional X are $M(T)\cdot N(T)=0$ or $Tp(T)=0$, $p(0)=0$, where $p(\lambda)$ is a polynomial. The equivalence may fail for infinite-dimensional X. The author sketches, as examples of (A) and (B), the mean ergodic theorem, invariant measures associated with a family of transformations of a set into itself, a situation including the existence and uniqueness of Haar measure, and the $L(-\infty,\infty)$ Wiener closure theorem for translations of a function. Actually the usual proofs of the mean ergodic theorem essentially introduce (B). For the measure examples the procedure is to introduce the Banach space of the bounded real or bounded continuous functions and to show that (A) implies the existence of a nontrivial linear functional vanishing on $N(G)$ and the measure sought is that involved in the known integral representation of such a functional.

Problems (A) and (B) can be framed in terms of projections. Thus P and $P'=I-P$ are to project X onto $M(G)$ and $N(G)$, respectively. Consider $\{T\}=G$ with (B) satisfied for each T and let $\{P(T)\}=\mathfrak{B}$ be the associated projections which we assume commute. With the usual interpretation $P(T)\wedge P(T')=P(T)P(T')$, $\mathfrak{B}$ is a Boolean algebra. To go from (B) to (A) involves the possibility of imbedding $\mathfrak{B}$ in a complete Boolean algebra. Suppose $\|P(T)\|\leqq C$, $T\in G$; then the author shows that if X is reflexive then such an embedding is possible. This embedding or completion can be carried out in several ways including the natural methods of Moore-Smith limits and the closure in the weak topology of the ring of operators on X. *D. G. Bourgin.*

James, Robert C. Orthogonality and linear functionals in normed linear spaces. Trans. Amer. Math. Soc. **61**, 265–292 (1947).

This paper contains a detailed study of orthogonality in normed linear spaces and its connections with linear functionals and hyperplanes. An element x is said to be orthogonal to y $(x\perp y)$ if and only if $\|x+ky\|\geqq\|x\|$ for all real numbers k. An element x_0 is orthogonal to at least one hyperplane through the origin. Each such hyperplane can be characterized as the set of null vectors of a linear functional $f(x)$ for which $f(x_0)=\|f\|\cdot\|x_0\|$. Given x and y there exists at least one pair of numbers a, b such that $x\perp ax+y$ and $bx+y\perp x$. The author relates orthogonality to the right and left norm derivatives $N_\pm(x;y)=\lim_{k\to\pm 0}(\|x+ky\|-\|x\|)/k$. It is shown that $x\perp ax+y$ if and only if $N_-(x;y)\leqq -a\|x\|\leqq N_+(x;y)$. It follows that an element x is orthogonal to only one hyperplane if and only if $N_-(x;y)=N_+(x;y)$ for all y. Similarly there is a unique number b such that $bx+y\perp x$ if and only if the space is strictly convex. Several characterizations of abstract Euclidean spaces are given. It is shown, for instance, that a normed linear space is an abstract Euclidean space if and only if $N_+(x;\|x\|y)=N_+(y;\|y\|x)$ for all elements x and y. The conditions under which there exist elements orthogonal to given linear subsets of a normed linear space are also investigated. *R. S. Phillips.*

James, Robert C. Inner products in normed linear spaces. Bull. Amer. Math. Soc. **53**, 559–566 (1947).

The author gives several characterizations of abstract Euclidean spaces of three or more dimensions in terms of orthogonality, hyperplanes and linear functionals. Given two elements of a normed linear space x and y, then x is said to be orthogonal to y $(x\perp y)$ if and only if $\|x+ky\|\geqq\|x\|$ for all real k. Orthogonality is symmetric if $x\perp y$ implies $y\perp x$; it is additive on the left if $x\perp z$ and $y\perp z$ imply $x+y\perp z$. Symmetry of orthogonality or left additivity are each necessary and sufficient conditions for a normed linear space to be Euclidean. The proofs of these theorems rely on the characterization given by S. Kakutani [Jap. J. Math. **16**, 93–97 (1939); these Rev. **1**, 146] and R. S. Phillips [Bull. Amer. Math. Soc. **46**, 930–933 (1940); these Rev. **2**, 220] which requires the existence of a projection of unit norm on each two dimensional subspace. One of the theorems states that a normed linear space is Euclidean if and only if, given $f(x)=\|f\|\,\|x\|$ and $g(y)=\|g\|\,\|y\|$ for linear functionals f and g and elements x and y, there exist numbers a and b such that $f(ax+by)+g(ax+by)=\|f+g\|\,\|ax+by\|$ and $ax+by\neq 0$. *R. S. Phillips* (Los Angeles, Calif.).

Grunblum, M. M. Sur la théorie des systèmes biorthogonaux. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 287–290 (1947).

Let X be a separable Banach space and $\{x_i\}$, $\{F_i\}$ a biorthogonal system in X, where $\{x_i\}$ is fundamental in X and $\{F_i\}$ is a total set of linear functionals. The author obtains conditions for $\{x_i\}$ to be a basis for X. For this purpose he introduces two auxiliary spaces E and E_*: E is the set of all $x\in X$ such that $\sum_1^\infty F_i(x)x_i$ is convergent, with the norm $\|x\|_*$ defined by $\|x\|_*=\sup\|\sum_1^n F_i(x)x_i\|$; E_* is the set of all $x\in X$ for which $\|x\|_*<\infty$, where $\|x\|_*$ is again the norm. Then E is always complete and E_* is complete if $\{F_i\}$ is fundamental in X (as when X is reflexive). For $\{x_i\}$ to be a basis in X it is necessary and sufficient that E coincides with E_* and sufficient that E is reflexive. A condition is also given for a reflexive space X which makes E reflexive. *B. Yood* (Ithaca, N. Y.).

Monna, A. F. Sur les espaces linéaires normés. I, II, III, IV. Nederl. Akad. Wetensch., Proc. **49**, 1045–1055, 1056–1062, 1134–1141, 1142–1152 = Indagationes Math. **8**, 643–653, 654–660, 682–689, 690–700 (1946).

Let K be a field complete with respect to a nontrivial non-Archimedean valuation $|\cdots|$ into the set of real numbers. Let E be a vector space over K with norm $\|\cdots\|$ such that (1) if $x \in E$, $x \neq 0$, then $\|x\| > 0$, (2) $\|ax\| = |a| \cdot \|x\|$, $a \in K$, (3) $\|x+y\| \leqq \max(\|x\|, \|y\|)$, (4) the set of real numbers $\|x\|$ has no limit point but 0. Part I contains conditions for E to be locally compact, and in particular it is shown that a locally compact space is finite-dimensional. It is proved also that a finite-dimensional subspace of any E is closed. In part II it is shown that the elements of a complete space may be expanded in series similar to the power series expansions valid for elements of K. On the basis of these expansions theorem 8 states a sufficient condition for two locally compact linear spaces to be topologically isomorphic; it seems to the reviewer, however, that this criterion is incorrect. Part III contains an analogue of the Hahn-Banach theorem and some other results similar to those in a previous paper by the author [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 308–321 (1943); these Rev. **8**, 32]. In part IV, a theory of projections in complete spaces E is developed, a projection on a closed linear subspace V being defined as a continuous linear operator P such that for $x \in E$, $Px \in V$, and $\|x-Px\| \leqq \|x-y\|$ for every $y \in V$. A projection P is shown to exist on every closed linear subspace V, and although P is not uniquely determined by V, it is shown to have many of the properties of projections in Hilbert space. Certain of the theorems in this series of papers are proved without the use of assumption (4) above. There is also some discussion of what can be said if the valuation of K (but not the norm in E) is allowed to be trivial. *I. S. Cohen* (Philadelphia, Pa.).

Plesner, A. I., and Rohlin, V. A. Spectral theory of linear operators. II. Uspehi Matem. Nauk (N.S.) **1**(11), no. 1, 71–191 (1946). (Russian)

[Part I, by Plesner alone, appeared in the same journal **9**, 3–125 (1941); these Rev. **3**, 210.] Part II comprises §§ 14–29, as follows: 14. Operator-valued measure. 15. Integrability with respect to spectral measure. 16. Functions of Hermitian operators. 17. Closed operators permutable with Hermitian operators. 18. Functions of permutable Hermitian operators. 19. Spectral analysis of unitary operators. 20. Closed operators. 21. Spectral analysis of normal operators. 22. Hellinger types. 23. Cyclic operators. 24. Orthogonal sums of Hermitian operators with pairwise independent maximal types. 25. Operators with simple spectra. 26. Multiple spectra. 27. The unitary invariants of Hermitian operators. 28. Generalized functions of a Hermitian operator. 29. The characterization of the functions of a Hermitian operator. There is an appendix which treats the Lebesgue-Stieltjes integral. This is a polished presentation of more or less standard material, so arranged as to include the case of nonseparable complex Euclidean spaces.

The discussion of unitary invariants in §§ 22–28, while basically the same as the known treatments (beginning with the original ones of Hellinger and Hahn), applies to the nonseparable case and is made particularly clear by systematic use of lattice-theoretic concepts. The family of finite mass-distributions on the infinite line, $-\infty < \lambda < \infty$, (or, what is the same thing, the family of the corresponding cumulative distribution functions ρ) falls into equivalence classes $\bar{\rho}$ if two distributions are regarded as equivalent whenever each has a density-function with respect to the other. The collection R of all such classes (called Hellinger types) is partially ordered by putting $\bar{\rho} \leqq \bar{\sigma}$ whenever ρ has a density function with respect to σ; and, indeed, R is found to be a countably-additive generalized Boolean algebra in which the relations $\bar{\rho}_\alpha \neq \bar{0}$, $\bar{\rho}_\alpha \cap \bar{\rho}_\beta = \bar{0}$ for $\alpha \neq \beta$, $\bar{\rho}_\alpha \leqq \bar{\rho}$ imply that the totality of elements $\bar{\rho}_\alpha$ is countable. Now, if A is a Hermitian (i.e., self-adjoint) operator in a fixed complex Euclidean space and $E(\lambda)$ is its resolution of the identity, a single vector or, more generally, a set of vectors $\{f_\alpha\}$ is said to be of type $\rho \neq 0$ (with respect to A) when $(E(\lambda)f_\alpha, E(\lambda)f_\beta) = \delta_{\alpha\beta}\rho(\lambda)$ for all α, β, λ. The maximal systems of type ρ all have the same cardinal number, called the multiplicity of ρ, which is a function $m_A(\bar{\rho})$ of the Hellinger type $\bar{\rho}$ alone. By virtue of the fact that $\bar{\rho}_n \neq \bar{0}$, $\bar{\rho}_m \cap \bar{\rho}_n = \bar{0}$ imply $m_A(\sum_{n=1}^{\infty} \bar{\rho}_n) = \min_n m_A(\bar{\rho}_n)$, it follows that the function m_A has a very simple structure: the algebra R is the direct sum of subalgebras on each of which m_A is constant; and $m_A(\bar{\rho})$ is equal to $\min_n m_A(\bar{\rho}_n)$, where $\{\bar{\rho}_n\}$ is the set of $\bar{\rho}$'s components, other than $\bar{0}$, in the subalgebras in question. The function m_A is then seen to be a characteristic unitary-invariant of A: in fact, $m_A(\bar{\rho}) = m_B(\bar{\rho})$ for all $\bar{\rho}$ if and only if A and B are unitary-equivalent. To define generalized functions of A, it is supposed that a complex-valued function $\Phi(\lambda, \bar{\rho})$ is defined for $-\infty < \lambda < +\infty$ and $\bar{\rho} \in R$ when $m_A(\bar{\rho}) \neq 0$, with the properties: $\Phi(\lambda, \bar{\rho})$ is a function of λ measurable with respect to ρ; if $\bar{\rho} \leqq \bar{\sigma}$ then $\Phi(\lambda, \bar{\rho}) = \Phi(\lambda, \bar{\sigma})$ except on a set of ρ-measure zero. Then there exists a unique closed linear operator B which is related to A in the following manner: if f is any vector and $M(f)$ the closed linear manifold generated by the vectors $E(\lambda)f$, $-\infty < \lambda < +\infty$, then $M(f)$ reduces both A and B, their respective components A_f and B_f in $M(f)$ being connected by the equation $B_f = \Phi(A_f, \bar{\rho})$, where $\rho(\lambda) = (E(\lambda)f, f)$. This operator is denoted by $B = \Phi(A)$. A necessary and sufficient condition that a closed linear operator be expressible in the form $\Phi(A)$ is that it commute with every bounded linear operator which commutes with A. *M. H. Stone* (Chicago, Ill.).

Plesner, A. I. Fundamental ideas of the spectral theory of Hermitian operators. Uspehi Matem. Nauk (N.S.) **1**(11), no. 1, 192–216 (1946). (Russian)

This is a purely expository paper, apparently written for the orientation of the reader who desires to study the immediately preceding paper [see the preceding review] without having ready access to its first part. *M. H. Stone.*

Marrot, Raymond. Extension d'un théorème de Toeplitz. C. R. Acad. Sci. Paris **224**, 1469 (1947).

Let H be a positive self-adjoint operator in Hilbert space $\mathfrak{H}$ and let us denote by $n(H)$ the greatest lower bound of (Hf, f) for f in the domain of H, $\|f\| = 1$. A theorem of Toeplitz asserts that, A being a (not necessarily self-adjoint) bounded operator, the conditions $n(A^*A) > 0$, $n(AA^*) > 0$ are necessary and sufficient for the existence of a bounded inverse A^{-1}. This theorem is shown to hold also for unbounded closed operators A, with domain everywhere dense in $\mathfrak{H}$. Separability of $\mathfrak{H}$ is not assumed. *B. de Sz. Nagy.*

Fichera, G. Sui funzionali continui con la metrica di Fréchet. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 174–177 (1947).

Consider the sphere S_R of radius R in a Hilbert space, a complete orthonormal set φ_i in this sphere, two arbitrarily selected functions φ, ψ in S_R, and their Fourier coefficients

$c_k=(\varphi, \varphi_k)$, $\gamma_k=(\psi, \varphi_k)$. Finally let

$$F(\varphi, \psi)=\sum_{k=1}^{\infty}|c_k-\gamma_k|/k!(1+|c_k-\gamma_k|)$$

be the Fréchet metric. Then Picone proved that S_R is closed and compact relative to this metric. Using this result, the author proves the following result. If $J(\varphi)$ is a functional, defined and continuous relative to the Fréchet metric over S_R, then the upper (lower) bound of J on the boundary of S_R is equal to the maximum (minimum) of J in all of S_R. This is an extension of a result by the author on quadratic functions [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 213–216 (1943); these Rev. **8**, 278, 709]. He also proves the following results. The upper and lower bounds of the polynomial $P(\varphi)=\sum_{r=1}^{n}\int K_r(x_1, \cdots, x_r)\varphi(x_1)\cdots\varphi(x_r)dx_1\cdots dx_r$ on the boundary of S_R are equal to the maximum and minimum of P in all of S_R; a necessary and sufficient condition that J, continuous relative to the Fréchet metric, be constant is that it be constant on the boundary of S_R.

H. H. Goldstine (Princeton, N. J.).

Hartman, Philip, and Wintner, Aurel. The (L^2)-space of relative measure. Proc. Nat. Acad. Sci. U. S. A. **33**, 128–132 (1947).

Let $(\bar{N}^p)$, $p\geqq 1$, denote the space of functions $f(x)$ on $0<x<\infty$ for which $\bar{M}\{|f|^p\}<\infty$, where

$$\bar{M}\{g\}=\limsup_{X\to\infty}X^{-1}\int_0^X g(x)dx, \qquad g\geqq 0.$$

With $\bar{M}\{|f-g|^p\}^{1/p}$ as distance, $(\bar{N}^p)$ is a linear metric space; the subspace N_p of functions for which $M\{|f|^p\}$ exists, where $M\{g\}=\lim_{X\to\infty}X^{-1}\int_0^X g(x)dx$, is not linear. By arguments due to Nalli and Besicovitch it follows that $(\bar{N}^p)$ and (N^p) are complete metric spaces. As has been pointed out to the reviewer by the authors, their simplification of Besicovitch's proof, which is incomplete in its present form, may be rectified by using for $k+1\leqq j\leqq i-1$ the inequality $|g_j-g_k|^2\leqq 2\{|g_j-g_i|^2+|g_i-g_k|^2\}$. The last formula line on page 130 then becomes correct if the last term is changed to $2\sum_{j=k}^{i}\sigma(n_j)$. [For a detailed study of $(\bar{N}^p)$ and other spaces see Bohr and Følner, Acta Math. **76**, 31–155 (1945); these Rev. **7**, 154, and Følner, Copenhagen thesis, 1944; these Rev. **8**, 151.]

B. Jessen (Copenhagen).

Arens, Richard. Representation of *-algebras. Duke Math. J. **14**, 269–282 (1947).

A theorem of Gelfand and Neumark [Rec. Math. [Mat. Sbornik] N.S. **12**(**54**), 197–213 (1943); these Rev. **5**, 147] is proved in a fashion which avoids the use of a lemma to which they refer and whose proof is not yet available, and which yields a somewhat more general result than theirs. The author's main result is that a Banach algebra A over the complex field with an adjoint operation and whose norm satisfies the inequality $k\|f\|\cdot\|f^*\|\leqq\|ff^*\|$, f being a general element of A and k being a fixed positive number, is algebraically isomorphic with the algebra of all complex-valued continuous function on a locally compact Hausdorff space X which "vanish at infinity"; furthermore, $k^n\|f\|\leqq\|f'\|_X\leqq\|f\|$, where $\|f'\|_X$ is the maximum of the absolute value of the continuous function f' corresponding to f, and $n=2$ or 20 according as A has or has not an identity. This is an improvement of the Gelfand-Neumark result in that they treat only the case in which A has an identity and $k=1$. A representation theorem for Banach algebras over the real field with an adjoint follows.

I. E. Segal (Chicago, Ill.).

***Köthe, Gottfried. Zusammenhänge zwischen der Funktionentheorie und der Theorie der Gleichungen mit unendlich vielen Unbekannten.** Ber. Math.-Tagung Tübingen 1946, pp. 92–94 (1947).

A short report (without proofs) of certain results dealing with λ- and λ^*-spaces. Let the denumerable set of "points" $a^{(k)}=\{a_1^{(k)}, a_2^{(k)}, \cdots\}$, $k=1, 2, \cdots$, be given sequences of complex numbers. Then λ^* is the complete space of all vectors $u=\{u_1, u_2, \cdots\}$ for which $\sum_i|a_i^{(k)}u_i|<\infty$, $k=1, 2, \cdots$; its dual space λ is the smallest complete space containing all the $a^{(k)}$. Strong and weak convergence are equivalent in a λ^*-space and a criterion can be given for this to be the case in a λ-space. The quotient space λ/a, of the closed subspaces of a λ-space, is complete, and to every weakly convergent sequence in λ/a there corresponds a weakly convergent representation sequence in λ. These results aid in the study of the homomorphic mappings of λ-spaces on λ-spaces. Earlier results of Pólya [Comment. Math. Helv. **11**, 234–252 (1939)] and Eidelheit [Studia Math. **6**, 104–111, 117–138 (1936)], dealing with certain functiontheoretic problems treated by means of systems of linear equations in infinitely many unknowns, are stated to be special cases of the present theory. Details are promised for a forthcoming article.

I. M. Sheffer (State College, Pa.).

Calculus of Variations

Berwald, L. Über Haars Verallgemeinerung des Lemmas von Du Bois-Reymond und verwandte Sätze. Acta Math. **79**, 39–49 (1947).

Using properties of solutions of a system of ordinary linear differential equations of the first order, the author gives a direct proof of the following form of the fundamental lemma of the calculus of variations. If the coefficients $b_{ik}(x)$, $a_{ik}(x)$ $(i, k=1, \cdots, n)$, of the differential forms $L_i(u)=a_{ik}(x)u_k'(x)+b_{ik}(x)u_k(x)$ are, respectively, continuous and of class C' on $[a, b]$: $a\leqq x\leqq b$, and $\|a_{ik}(x)\|$ is nonsingular on this interval, while $f_i(x)$ $(i=1, \cdots, n)$, are continuous functions such that

$$\int_a^b f_i(x)L_i(\eta)dx=0 \tag{1}$$

for arbitrary functions $\eta_i(x)$ of class C' on $[a, b]$ satisfying $\eta_i(a)=0=\eta_i(b)$, then the $f_i(x)$ are of class C' and satisfy $[f_i(x)a_{ik}(x)]'-f_i(x)b_{ik}(x)=0$. For the corresponding result for a single linear differential form of the nth order the author presents two methods of proof. One method, which attempts to reduce the problem to that mentioned above, is incorrect, since for the integral (1) of the reduced problem the functions $\eta_i(x)$ are required to satisfy auxiliary differential equations. The steps of the second method parallel those in the initial treatment; the same method has been used recently by the reviewer [Duke Math. J. **12**, 685–694 (1945); these Rev. **7**, 419] to prove a result more general than the one derived in this paper. In conclusion, there is given an extension involving iterates of an nth order linear differential form.

W. T. Reid (Evanston, Ill.).

Gheorghiu, Octavian Em. Sur la variation asynchrone d'une intégrale simple. Bull. Sci. École Polytech. Timişoara **12**, 163–166 (1946).

Consider the problem of minimizing the integral

$$I=\int_a^b L(x, x', \cdots, x^{(m)}; t)dt,$$

in which $x=(x_1, \cdots, x_n)$. Variation of each dependent variable x_i leads to an Euler equation $L_i=0$, where

$$L_i=\sum_{k=0}^{m}(-1)^k D^k L_{x_i^{(k)}}, \quad D=d/dt.$$

Variation of the independent variable t leads to an equation $L_0=0$ which is too complicated to reproduce here. When $m=1$, it is known that $L_0=-\sum_{i=1}^{n}x_i' L_i$. The author shows that this relation is still true when $m>1$.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Federighi, Urbano. Sul problema di Newton. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 179–185 (1946).

The methods of Tonelli are used to prove the existence of an absolute minimum for an integral

$$(1) \qquad \int_{x_1}^{x_2} yy'\{\phi(x, y)+Ny'\}dx$$

in the class of monotone nondecreasing absolutely continuous arcs $y=y(x)$ joining two fixed points P_1: (x_1, y_1) and P_2: (x_2, y_2), where $x_1<x_2$, $y_1<y_2$. For the integrand of (1) it is supposed that N is a positive constant and that in a certain region A_1 containing the rectangle with sides parallel to the coordinate axes and having P_1 and P_2 as opposite vertices the function $\phi(x, y)$ is of class C'', with $\phi_x\leqq 0$, $\phi_{xy}<0$ in the part of A_1 where $y\geqq y_1$, and $\phi_x\geqq 0$ in the part where $y<y_1$. The author states that such a variational problem appears as a "problem of Newton" for the determination of the surface of revolution encountering least resistance in certain cases of motion at supersonic velocity.

W. T. Reid (Evanston, Ill.).

Giuliano, Landolino. La variazione prima nei problemi di Lagrange. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 135–144 (1947).

The author gives a proof of a multiplier rule for the problem of Lagrange in the case when the differential equations of the side conditions are solved for the derivatives of some of the variables, denoted by u_i. The end values of the variables u_i are prescribed only at the first end point of the arc. The problem is considered in parametric form, and admissible curves are those which are rectifiable. More general cases have been treated by the reviewer [Amer. J. Math. **53**, 547–554 (1931)] and by McShane [Duke Math. J. **7**, 1–27 (1940); these Rev. **2**, 226]. *L. M. Graves.*

Shiffman, Max. Differentiability and analyticity of solutions of double integral variational problems. Ann. of Math. (2) **48**, 274–284 (1947).

This paper presents a new method for establishing differentiability of stationary surfaces for double integrals. To exhibit the essentials more clearly, only integrals $\iint F(p, q)dxdy$ are considered. If Z makes this stationary, the difference quotient $Z^{(h)}=\{z(x+h, y)-z(x, y)\}/h$ makes a certain integral stationary; the integrand of this integral is quadratic in the first partials of Z^h. For quadratic integrands, a theorem related to Green's theorem is established; it yields an estimate of the integral over every circle whose closure is interior to the domain of integration, provided the integral is stationary. Applying this to the $Z^{(h)}$ yields a uniform bound for their Dirichlet integrals over every such circle, whence it follows that the $Z^{(h)}$ are equicontinuous and uniformly convergent on the circle, and so Z_x is continuous. To establish the existence and continuity of higher derivatives, the author uses Morrey's growth theorems and Hopf's theorem on functions whose derivatives satisfy a Hölder condition. He indicates that extension of the method of this paper may yield higher derivatives directly.

E. J. McShane (Charlottesville, Va.).

Lepage, Th. H. Sur une classe d'équations aux dérivées partielles du second ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 140–151 (1947).

Let $P=(p_{ij})$ be a symmetric matrix ($p_{ij}=\partial^2 z/\partial x^i \partial x^j$, $i, j=1, \cdots, n$), E the $n\times n$ unit matrix, and let $F(p_{ij})$ be a function linear in the determinants of order n of the rectangular matrix (E, P) with coefficients analytic functions of the arguments x_i, z, $\partial z/\partial x_i$. In an earlier paper [Bull. Soc. Roy. Sci. Liége **15**, 21–31 (1946); these Rev. **8**, 499] the author associated with each $F(p_{ij})$ a unique symbolic (alternée) form which we represent by Λ. We shall denote by (T) the subclass of functions $F(p_{ij})$ such that the equation $F(p_{ij})=0$ results from the vanishing of the first variation of a multiple integral. The question of whether or not $F(p_{ij})$ belongs to the class (T) is discussed in the present paper in terms of symbolic forms. The results obtained are as follows. If $F(p_{ij})$ belongs to class (T), there exists a function $f(x_i, z, p_i)\neq 0$ such that $f\Lambda w$ is integrable ($w=dz-p_1dx_1-\cdots-p_ndx_n$; $p_i=\partial z/\partial x_i$). If, in addition to $f\Lambda w$ being integrable, Λw is not singular, then $F(p_{ij})$ belongs to class (T). *F. G. Dressel* (Durham, N. C.).

Theory of Probability

Mises, R. Ueber Aufteilungs- und Besetzungs-Wahrscheinlichkeiten. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **27**, 21 pp. (1939). (German. Russian summary)

The author introduces his problem by considering the probability that out of a group of k people, three will have the same birthday. He assumes that the chance that a particular person will have a specific birthday is $1/n$, where $n=365$. If x_i denotes the number of days on which there are exactly i of the k birthdays for a given group of people, then the numbers $x_0, \cdots, x_k$ constitute a frequency distribution satisfying the relations $\sum x_i=n$, $\sum ix_i=k$. The author computes the probability of obtaining a given distribution and also finds the most probable distribution. He shows that the expected values of $x_0, \cdots, x_k$ approach the normal distribution if k becomes infinite while n remains fixed whereas they approach the Poisson distribution if n and k simultaneously become infinite in such a manner that k/n^2 approaches zero. The above frequency distribution arises in connection with quantum theory and the results are applicable there. *A. H. Copeland* (Ann Arbor, Mich.).

Fréchet, M. The general relation between the mean and the mode for a discontinuous variate. Ann. Math. Statistics **18**, 290–293 (1947).

If R is a random variable which takes on only nonnegative integer values, and if α is a positive integer which is an upper bound for the mean of the distribution, then $\alpha(\alpha+3)/2$ is an upper bound for the mode. The bound cannot be improved. This result is related to some of the author's previous work relating means, modes and equiprobable values. *G. W. Brown* (Ames, Iowa).

Smirnov, N. V. Sur un critère de symétrie de la loi de distribution d'une variable aléatoire. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 11–14 (1947).

Let $X_1, \cdots, X_n$ be mutually independent random variables with a common distribution function $F(x)$ which is continuous and symmetric with respect to $x=0$. Let $N^+(x)$ and $N^-(x)$ be the numbers of X_k contained in the intervals $(0, x)$ and $(-x, 0)$, respectively. Put $t_n=\max\{N^+(x)-N^-(x)\}$ and $t_n^*=\max\{|N^+(x)-N^-(x)|\}$. The random variables t_n and t_n^* are integer valued and their distributions are independent of $F(x)$. If the X_k are rearranged in increasing order of $|X_k|$, it is easily seen that $\Pr\{t_n \geqq \nu\}$ equals the probability of a ruin in n trials or less in the classical coin game when the adversary is infinitely rich and the player has an initial capital ν. The distribution of t_n, and similarly of t_n^*, is therefore well known. The result can be used for testing the statistical hypothesis that an unknown distribution is symmetric. *W. Feller* (Ithaca, N. Y.).

***Schulz, Günther. Das Summenproblem bei mehrdimensionalen arithmetischen Wahrscheinlichkeitsverteilungen.** Ber. Math.-Tagung Tübingen 1946, pp. 131–134 (1947).

Let $t_1, t_2, \cdots$ be a sequence of independent chance vectors with identical distributions, each vector with k components, thus: $t_i=(t_{i1}, \cdots, t_{ik})$. For any j the probability is one that t_{ij} is a bounded integer. Let $a_j=Et_{ij}$, and the matrix $\{\sigma(t_{ij}t_{il})\}=\{a_{jl}\}^{-1}$. Sufficient conditions are given that

$$\lim_{n\to\infty} n^{k/2}P\left\{\sum_{i=1}^{n} t_{ij}=na_j+n^{\frac{1}{2}}u_j, \quad j=1, 2, \cdots, k\right\}$$
$$=\{|a_{jl}|/(2\pi)^k\}^{\frac{1}{2}}\exp\left[-\tfrac{1}{2}\{\sum_{j,l} a_{jl}u_ju_l\}\right].$$

J. Wolfowitz (New York, N. Y.).

Kac, Mark. Random walk and the theory of Brownian motion. Amer. Math. Monthly **54**, 369–391 (1947).

Exposé d'ensemble sur le mouvement brownien sur une droite, considéré dans le cas discret, comme une promenade au hasard. Au lieu de la probabilité élémentaire $P(x_0|x, t)dx$ qu'à l'instant t la particule soit dans $(x, x+dx)$ si elle se trouvait en x_0 à l'instant $t=0$, l'auteur étudie la probabilité $P(n|m; s)$ qu'elle soit en $m\Delta$ à l'instant $s\tau$ (Δ et τ fixes) si elle se trouvait en $n\Delta$ à l'instant 0. En résolvant par la méthode matricielle les équations aux différences correspondantes, il détermine les valeurs explicites de cette probabilité dans les cas où la particule est, soit libre, soit dans un champ de forces uniforme avec réflexion à l'origine, soit élastiquement liée. Par passage à la limite il retrouve les expressions des $P(x_0|x, t)$ dues respectivement à Einstein, Smoluchowski et Rayleigh. Il examine ensuite le modèle d'Ehrenfest, qui permet de discuter l'antinomie apparente entre l'irréversibilité physique et la récurrence, et donne pour ce modèle l'expression de la durée moyenne de récurrence en utilisant la méthode des fonctions génératrices. [L'auteur signale les errata suivants: pp. 377, 378 lire $\sin m\Theta_k \cos\Theta_k$ au lieu de $\sin m\Theta_k$ et lire $\sinh m\theta_0 \cosh\theta_0$ au lieu de $\sinh m\theta_0$; p. 389 lire $|z|<1$ au lieu de $|z|>1$.] *M. Loève* (London).

Slutsky, E. Quelques propositions sur la théorie des fonctions aléatoires. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **31**, 15 pp. (1939). (Russian. French summary)

Five theorems are proved. Let $y(t)$ be a stochastic process with a continuous parameter t. It is said to be stochastically continuous (s.c.) at the point t if, for any positive ϵ, η, there exists a positive τ such that $P\{|y(t)-y(t')|<\epsilon\}>1-\eta$ whenever $|t-t'|<\tau$. Uniform stochastic continuity is defined in the obvious way. Theorem 1 states essentially that if the process is s.c. in the interval $[a, b]$, it is uniformly s.c. in the interval. Two processes $y(t)$ and $z(t)$ are called stochastically equivalent (s.e.) if, for every t, $P\{y(t)=z(t)\}=1$. Theorem 3 states that if $y(t)$ is s.c. in $[a, b]$, it is s.e. to a process $z(t)$ whose sample functions are, with probability one, almost everywhere in $[a, b]$ limits of sequences of continuous functions. Theorem 4, said to have been originally given by Kolmogoroff in 1934, states that if, in the interval $[a, b]$, $E|y(t)-y(t')|^m<C|t-t'|^\alpha$, where $m>0$, $\alpha>1$, then $y(t)$ is s.e. to a process $z(t)$, continuous in $[a, b]$. A regular process $y(t)$ is one which possesses finite and continuous (in t) moments of all positive orders less than p, where p is positive and can be infinite. Theorem 2. Suppose a process $y(t)$, $a\leqq t\leqq b$, possesses finite even positive moments of order less than p, where p is either $+\infty$ or such that, for all $\epsilon\geqq 0$, $E|y(t)|^{p+\epsilon}=\infty$. Then the continuity (in t_1, t_2) of $E[y(t_1)]^\lambda[y(t_2)]^\lambda$, where λ is positive, less than $p/2$ and of the form $(2m+1)/(2n+1)$, $m, n=0, 1, 2, \cdots$, is necessary and sufficient that $y(t)$ is (A) s.c., (B) regular. Theorem 5. Let the stochastic process $y(t)$, $a\leqq t\leqq b$, be such that (1) $Ey(t)^2$ is bounded and $\sigma^2(t)>0$ in $[a, b]$, (2) $Ey(t)^2$ and $\sigma(t)$ are differentiable and (3) $\rho(t, \tau)$ has bounded derivatives of the fourth order, where $\rho(t, \tau)$ is the correlation coefficient between $y(t+\tau/2)$ and $y(t-\tau/2)$. Then there is a s.c. process $z(t)$, s.e. to $y(t)$, which, with probability one, is differentiable everywhere in $a\leqq t\leqq b$. [There are a few typographic errors. The French résumé is inadequate.] *J. Wolfowitz.*

Kolmogoroff, A. N., and Dmitriev, N. A. Branching stochastic processes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 5–8 (1947).

Suppose that objects are divided into n types. There are transitions in which each object goes into one or more objects of each of the n types, in accordance with some probability law. The transitions are independent of each other and of past transitions. If $\alpha(t)$ is the vector whose kth component is the number of objects of type k at time t, the α process is then a Markov process, and the standard theory of Markov processes can be applied, but it is simpler to use special methods adapted to this special case. When $n=1$, the process becomes the birth process studied by many authors [see, for example, R. A. Fisher, The Genetical Theory of Natural Selection, Oxford University Press, 1930]. The authors find a functional equation and a differential equation for the generating function of the $\alpha(t)$ distribution, and show how simply the differential equation can be used to find the transition probabilities of a simple birth process examined by Arley [On the Theory of Stochastic Processes and their Application to the Theory of Cosmic Radiation, Copenhagen thesis, 1943; these Rev. **7**, 209].

J. L. Doob (Urbana, Ill.).

Levinson, Norman. A heuristic exposition of Wiener's mathematical theory of prediction and filtering. J. Math. Phys. Mass. Inst. Tech. **26**, 110–119 (1947).

Expository paper. *J. L. Doob* (Urbana, Ill.).

Kampé de Fériet, Joseph. Sur une représentation des fonctions aléatoires. C. R. Acad. Sci. Paris **225**, 37–38 (1947).

Let $f(\omega)$ be a measurable function defined on a measure space (of total measure 1). Then if $\{T_\nu\}$ is a family of

measurability preserving transformations of ω space into itself, with $-\infty<s<\infty$, $T_{s+t}=T_s\cdot T_t$, it is noted that the definition $X(t)=f(T_t\omega)$ can be used to define the chance variables $X(t)$ of a stochastic process. The author compares this definition with the also current definition in terms of measure in function space. [Cf. for a full comparison of the two approaches Doob and Ambrose, Ann. of Math. (2) 41, 737–745 (1940); these Rev. 2, 108.] *J. L. Doob.*

Kampé de Fériet, J. Sur la moyenne des mesures, dans un écoulement turbulent, des anémomètres dont les indications sont indépendantes de la direction de la vitesse. La Météorologie (4) 1, 133–143 (1946).

Almost all anemometers measure the component of fluid velocity on a certain plane (rather than on a certain line). The author shows how to take account of this fact in finding the true mean speed, correcting for the deviations caused by turbulence. In particular it is shown how the anemometer reading is related to the mean velocity if the turbulent velocity vector is normally distributed. *J. L. Doob.*

Bass, Jean. Sur le corpuscule aléatoire à masse aléatoire. C. R. Acad. Sci. Paris 225, 38–40 (1947).

Let $X(t)$ be a chance three-dimensional vector, with derivative $U(t)$. Let F be the six-dimensional probability density of distribution of (X, U) at time t. It is supposed that there is a linear operator A acting on (X, U) functions for which $\partial F/\partial t=(A-\sum_i u_i\partial/\partial x_i)F$ and that, for fixed X, there is an adjoint operator A^* acting on U functions such that $\int\psi AF du_1du_2du_3=\int FA^*\psi du_1du_2du_3$. The fact that X has derivative U implies that expectation $A^*(1)=0$. From this is derived a general transfer equation. The treatment is then extended to the chance motion of a particle of variable mass, in particular to the case of the special theory of relativity in which the mass depends in a simple way on the velocity. [For a general review of this subject see Bass, Revue Sci. 83, 3–20 (1945); these Rev. 7, 460.]

J. L. Doob (Urbana, Ill.).

***Tchen, Chan-Mou. Mean Value and Correlation Problems Connected with the Motion of Small Particles Suspended in a Turbulent Fluid.** Martinus Nijhoff, The Hague, 1947. vii+125 pp. (English. Dutch summary)

The author treats diffusion problems as governed by the differential equations for the transition probabilities of a Markov process, derived by Kolmogoroff, but is apparently unaware of the special character of Markov processes and in general of modern developments in the theory of stochastic processes. *J. L. Doob* (Urbana, Ill.).

Dedebant, G., et Wehrlé, Ph. La mécanique des fluides turbulents fondée sur des concepts statistiques. Thalès 4, 151–167 (1940).

Expository lecture.

Ascoli, Guido. Sopra una valutazione asintotica che si presenta nella teoria probabilistica dei contatori di corpuscoli. Ricerca Sci. 17, 611–616 (1947).

New derivation of the asymptotic expansions for the mean and the variance of the number of registrations in a counter given by G. Lovera [Ricerca Sci. 17, 223–228 (1947); these Rev. 8, 523]. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

***Thurstone, L. L. Multiple-Factor Analysis. A Development and Expansion of "The Vectors of Mind."** The University of Chicago Press, Chicago, Ill., 1947. xix+535 pp. $7.50.

The scores obtained by an individual in performing several tasks can be considered as a vector; this is the prototype of a factorial situation. This vector can be considered as the sum of two vectors, one representing errors and skills involved in only one task; it is the main purpose of factor analysis to draw inferences concerning the other. The author has taken the lead in developing the theory and methods of this field for many years. Statistical theories have been developed for the case where the original coordinates of the vectors (=tasks) have no invariant significance. The author presents, in passing, a well and carefully reasoned argument for statistical developments in which this is not the case and where "simple structure" is to be expected.

A mathematical introduction of 50 pages covers the essentials of real vector and matrix theory, without proof. The body of the work is concerned with the background, computation, and interpretation of successively more complicated cases. This is the most thorough work in such a field which the reviewer has seen.

One recurrent mathematical error should be noted. The author states at several points that a set of vectors making angles of 90° or less with each other lies in a cone of half-angle 45°. (The positive coordinate directions in 3-space are a counter-example.) *J. W. Tukey* (Princeton, N. J.).

Wherry, Robert J. Multiple bi-serial and multiple point bi-serial correlation. Psychometrika 12, 189–195 (1947).

Van Boven, Alice. A modified Aitken pivotal condensation method for partial regression and multiple correlation. Psychometrika 12, 127–133 (1947).

The unit matrices used by Aitken are eliminated. Each of the successive matrices is reduced by the same rule. Checking is facilitated. *E. Bodewig* (The Hague).

Oberg, E. N. Approximate formulas for the radii of circles which include a specified fraction of a normal bivariate distribution. Ann. Math. Statistics 18, 442–447 (1947).

Given the normal uncorrelated bivariate error function, means zero and standard deviations σ_x, σ_y, the author presents three approximate formulas for the radii of circles, centers at the origin, which include a prescribed proportion p of errors. Tables are given to show the accuracy of the approximations for $.5\leqq\sigma_x/\sigma_y\leqq1$. The problem has been treated differently by D. Martin [Philos. Mag. (7) 37, 636–639 (1946); these Rev. 8, 523]. *L. A. Aroian.*

Leverett, Hollis M. Table of mean deviates for various portions of the unit normal distribution. Psychometrika 12, 141–152 (1947).

Table of the mean value of x, taken over the part of a standard normal distribution between the $100p$ and $100q$ percent points for $p\leqq q=0.00(.01)1.00$ (entry with $1-q$ and $1-p$ may be necessary). *J. W. Tukey* (Princeton, N. J.).

Francis, V. J. On the distribution of the sum of n sample values drawn from a truncated normal population. Suppl. J. Roy. Statist. Soc. 8, 223–232 (1946).

Tables are computed of the cumulative distribution function of the sum of several independent observations from a

normal distribution truncated at one end. An illustration of the industrial use of these tables is given. *J. Wolfowitz.*

Richardson, J. T. A table of Lagrangian coefficients for logarithmic interpolation of standard statistical tables to obtain other probability levels. Suppl. J. Roy. Statist. Soc. 8, 212–215 (1946).

It was found by Simaika [Biometrika 32, 263–276 (1942); these Rev. 4, 20] that interpolation for probability levels of distribution proceeded well and simply by polynomial interpolation in the logarithm of the probability. The present paper tabulates the interpolation coefficients for quadratic interpolation in $\log k$ for $x=35(1)75$ on 100, 50, 25, for $x=70(1)110(2)190$ on 250, 100, 50, and for $x=175(5)375$ on 500, 250, 100, where, up to a common factor, x is the desired probability and the three numbers are the known probability levels. (E.g., given 5%, 2.5%, 1% points to find the 3.2% point: use $x=320$.) *J. W. Tukey.*

Morse, Anthony P., and Grubbs, Frank E. The estimation of dispersion from differences. Ann. Math. Statistics 18, 194–214 (1947).

Let $\{x_i\}$ be an ordered sequence (uniformly in time) of n observations, with $x_i=\eta_i+\epsilon_i$, where $E(x_i)=\eta_i=f(t_i)$ and $\{\epsilon_i\}$ are normally and independently distributed with mean 0 and variance σ^2. This paper discusses the properties of backward differences of order p. The quantity

$$\delta^2_{n,p}=\frac{1}{\binom{2p}{p}(n-p)}\sum_{i=p+1}^{n}(\Delta^p\epsilon_i)^2$$

is an unbiased estimate of σ^2. The corresponding quantity for the $\{x_i\}$ is $d^2_{n,p}$ and for the η_i is $\nu^2_{n,p}$, with $E(d^2)=\sigma^2+\nu^2$. An upper bound is determined for ν^2 in terms of the average value of the squared pth derivative of f. Some values of this upper bound are given for $f(t)=\sin t$, for which ν^2 becomes quite small if $n>7$ and $p>2$.

The efficiency W of δ^2 relative to $s^2=\sum(x_i-\bar{x})^2/(n-1)$ as an estimate of σ^2 is tabulated for $p=1(1)10$ and $n=2(1)40(2)58$, various n up to 2314, and $n=\infty$, where $W=\text{var}\,(s^2)/\text{var}\,(\delta^2)$. It is also shown that $\text{var}\,(d^2)\leqq\text{var}\,(\delta^2)+4\nu^2\sigma^2$. It should be noted that W is actually asymptotic efficiency, but is quite generally used for small samples; δ^2 is about $\frac{3}{4}$ efficient for $p=1$, $\frac{1}{2}$ for $p=2$, and decreasing to $\frac{1}{4}$ for $p=10$. A table of approximate standard errors in estimating σ from differences is also given, assuming p is large enough to make ν^2 negligible. Estimation of σ^2 from differences could be quite useful if the form of $f(t)$ were unknown; and even if $f(t)$ could be satisfactorily estimated by least squares, differencing might be useful because of the saving in computing time for large n. *R. L. Anderson* (Raleigh, N. C.).

Schilling, Walter. A frequency distribution represented as the sum of two Poisson distributions. J. Amer. Statist. Assoc. 42, 407–424 (1947).

The author discusses extensively χ^2 tests of the hypotheses that an observed distribution corresponds to (1) a Poisson distribution, (2) a weighted average of two Poisson distributions. Applications are made to the examples given by F. Thorndike [Bell System Tech. J. 5, 604–624 (1926)]. *W. Feller* (Ithaca, N. Y.).

Scheffé, Henry. The relation of control charts to analysis of variance and chi-square tests. J. Amer. Statist. Assoc. 42, 425–431 (1947).

Walsh, John E. An extension to two populations of an analogue of Student's t-test using the sample range. Ann. Math. Statistics 18, 280–285 (1947).

The variant of the t statistic discussed by Daly [same Ann. 17, 71–74 (1946); these Rev. 7, 464] and Lord [Biometrika 34, 41–67 (1947); these Rev. 8, 394] involves use of the sample range to estimate the unknown standard deviation. This statistic is used to develop tests of hypotheses on the means of normal populations against one-sided alternatives. The new statistic is easier to compute than Student's t, and the power of the tests described compares favorably with those based on Student's t. *J. Wolfowitz.*

☆**Cansado Maceda, E. Cumulants of Fisher's z.** Revista Mat. Hisp.-Amer. (4) 7, 87–89 (1947). (Spanish)

Another computation of the first two cumulants of Fisher's z. [A more complete study by Wishart, Biometrika 34, 170–178 (1947); these Rev. 8, 474, was anticipated by Aroian, Ann. Math. Statistics 12, 429–448 (1941); these Rev. 3, 175.] Comparison of results yields the formula

$$\Psi(x)-\Psi(y)=\frac{x-y}{xy}\,{}_3F_2(x, y, 1; x+1, y+1; 1),$$

where Ψ is the digamma function (the derivative of $\log\Gamma(x)$). *J. W. Tukey* (Princeton, N. J.).

Aroian, Leo A. The probability function of the product of two normally distributed variables. Ann. Math. Statistics 18, 265–271 (1947).

Let x, y be normal with means ρ_1, ρ_2, unit variances, and correlation coefficient r. The author investigates the (asymptotically normal) distribution of the product xy for large ρ_1, ρ_2, approximating it by a Pearson type III distribution and a Gram-Charlier series and checking the accuracy by numerical integration. [Cf. Craig, same Ann. 7, 1–15 (1936).] *G. Elfving* (Helsingfors).

Elfving, G. A simple method of deducing certain distributions connected with multivariate sampling. Skand. Aktuarietidskr. 30, 56–74 (1947).

By using the comparatively easy ordinary regression theory, the author derives the following well-known results in normal multivariate sampling theory: the probability laws of the regression coefficients, the residual variances, the partial and multiple correlation coefficients, the generalized variance, and Hotelling's T^2 statistic. He also derives Wishart's distribution and proves Bartlett's decomposition theorem. The methods employed are similar to those of Lindblom [Skand. Aktuarietidskr. 29, 12–29 (1946); these Rev. 8, 42]. *L. A. Aroian* (New York, N. Y.).

Dumas, Maurice. Sur une loi de probabilité à priori conduisant aux arguments fiduciaires de Fisher. Revue Sci. 85, 3–18 (1947).

Let A and H be, respectively, the mean and the precision (reciprocal of the standard deviation) of a normal distribution. The author considers A and H as chance variables with an unknown a priori distribution, and examines a number of a priori distributions. He concludes that if, at $A=a$, $H=h$, the a priori probability density is k/h, where k is a constant, some "remarkable" results are obtained. Among these are expressions which, he says, lead to Fisher's fiducial argument. Whether this is so can be said only by one who claims to understand the latter. It is presumed that A ranges from $-\infty$ to $+\infty$ and H ranges from 0 to $+\infty$.

The reviewer's statement that only the first two cumulants are given was based on a misprint in the paper. In the formula quoted in the review, $y-x$ should be halves of positive integers. – Errata p. 735

In spite of this, the author is able to determine k, at least to his own satisfaction. *J. Wolfowitz* (New York, N. Y.).

Anscombe, F. J. Linear sequential rectifying inspection for controlling fraction defective. Suppl. J. Roy. Statist. Soc. **8**, 216–222 (1946).

Rectifying lot inspection schemes are considered where the defective units found are replaced by nondefective units and the lot is always accepted. The author gives an adaptation of the sequential probability ratio sampling plan to rectifying inspection which satisfies the condition that the probability is at most a given ϵ that the number of defectives in the lot after inspection will exceed a certain tolerance limit. Comparison of this sequential method with the Dodge-Romig scheme is made. *A. Wald* (New York, N. Y.).

Wolfowitz, J. The efficiency of sequential estimates and Wald's equation for sequential processes. Ann. Math. Statistics **18**, 215–230 (1947).

Let X be a chance variable with the distribution function $f(x, \theta)$, and let Ω be the space of points $\omega = x_1, x_2, \cdots$, where $x_1, x_2, \cdots$ is an infinite sequence of observations of X. Furthermore let there be given an infinite sequence of Borel measurable functions $\varphi_1(x_1), \varphi_2(x_1, x_2), \cdots, \varphi_j(x_1, \cdots, x_j), \cdots$ defined in Ω, such that each takes on only the values zero and one. It is assumed that, for every admissible value of θ, it is almost certain that at least one of the functions $\varphi_1, \varphi_2, \cdots$ takes the value one. Let $n(\omega)$ be the smallest integer at which this occurs. Let θ^* be an estimate of θ whose bias is $b(\theta)$. It is proved, subject to certain regularity conditions, that $\sigma^2(\theta^*) \geqq (1+db/d\theta)^2[EnE(\partial \log f/\partial\theta)^2]^{-1}$. Let the parameter θ consist of l components $\theta_1, \cdots, \theta_l$, for which there are given the unbiased estimates $\theta_1^*, \cdots, \theta_l^*$. It is proved ($n$ is still supposed to be a chance variable in the sense above) that the concentration ellipsoid of $\theta_1^*, \cdots, \theta_l^*$ always contains the ellipsoid

$$\sum_{i,j} \mu'_{ij}(k_i - \theta_i)(k_j - \theta_j) = l + 2,$$

where

$$\mu'_{ij} = EnE\left(\frac{\partial \log f}{\partial \theta_i} \frac{\partial \log f}{\partial \theta_j}\right),$$

as a generalization of Cramér's result for constant n [Skand. Aktuarietidsskr. **29**, 85–94 (1946); these Rev. **8**, 163]. In the last half of the paper the author gives a new proof of Wald's equation $E(Z_n) = EnEX$, where $Z_n = \sum_{i=1}^{n} x_i$, and states various new results in connection with this.

K. R. Buch (Copenhagen).

Wise, M. E. The use of the negative binomial distribution in an industrial sampling problem. Suppl. J. Roy. Statist. Soc. **8**, 202–211 (1946).

The author uses the negative binomial to describe the distribution of pinholes in reels of enamelled wire and supplies rejection rules based on the mean number as well as the maximum number of pinholes in samples of several lengths of wire. Criteria are calculated for 95% tests with selected sample sizes, and a sequential acceptance rule is suggested, without calculating the probabilities associated with the overall procedure. A mathematical appendix presents a convenient method for solving the maximum likelihood equations, an adaptation of Cochran's method [Ann. Math. Statistics **11**, 93–95 (1940); these Rev. **1**, 249] for solving the incomplete beta function equation $I_k(Nm, Nk) = \theta$, where m is unknown and θ is small, and an approximation to the incomplete beta function by means of a series of incomplete gamma functions. *G. W. Brown.*

Kosambi, D. D. An extension of the least-squares method for statistical estimation. Ann. Eugenics **13**, 257–261 (1947).

Let $f_i(x_1, \cdots, x_t) = 0$, $i = 1, \cdots, n$, $n > t$, be a set of (in general) inconsistent equations in $x_1, \cdots, x_t$. The author proposes as a "general principle" to use the point $(x_1, \cdots, x_t)$ such that the sum of the squares of its perpendicular distances to the surfaces $f_i = 0$ is a minimum. No probability set-up is indicated, and the merits of this procedure remain to be stated and proved. Most of the paper is devoted to the algebra of a given problem in genetics. *J. Wolfowitz.*

Marcantoni, Alessandro. Saggio di un'applicazione del calcolo delle matrici alla teoria degli errori. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 252–270 (1946).

The fundamentals of the least square theory of errors are presented in matrix notation. Particular attention is devoted to the computation of the variances and covariances of the variables. They are easily expressed in terms of the elements of the inverse of the normal matrix. *W. Feller.*

Mathematical Biology

***Luneburg, Rudolf K. Mathematical Analysis of Binocular Vision.** Princeton University Press, Princeton, N. J., 1947. vi+104 pp. $2.50.

On the basis of certain hypotheses of a psychological character, the author reduces the study of space relations inferred from binocular vision to problems in non-Euclidean geometry, with a metric implied by the sensations of binocular vision. The work is given in considerable detail, with application to problems of visual interpretation, but only the basic train of thought will be discussed in this review. From the complicated set of head and eye movements which might be used in viewing a set of object points, the author picks out two: fixed head (FH) and moving head (MH). In FH, the line joining the eyes (R and L) remains fixed and horizontal and the axes of the eyes are directed to an object point P. In MH, the line RL remains horizontal, but the head is turned about a vertical axis until the convergence of the two eyes is symmetric. Each such method of observation attaches to each point P of space a set of three physiological coordinates. In FH they are γ = angle RPL, $\phi = \frac{1}{2}(RLP \ominus LRP)$, θ = inclination of plane PRL to horizontal. In MH they are γ^* = angle RPL, ϕ^* = inclination to shoulder-line of line from midpoint of RL to P, θ^* = inclination of plane PRL to horizontal. Different sensations are produced by a given configuration of points according as they are observed FH or MH. In the case of two different configurations, one viewed FH and the other MH, the following strong hypothesis is made. If the points of the two configurations can be put into correspondence so that for corresponding points $\gamma = \gamma^*$, $\phi = \phi^*$, $\theta = \theta^*$, then the observer is led to an identical sensation for the two configurations and he interprets both configurations as identical. If a line-element PQ is observed, either FH or MH, the author assumes that there is in each case a definite sensation of size and that the square of this size is a quadratic differential form ($D(PQ)$ and $D^*(PQ)$, respectively).

If two corresponding elements are observed, it follows from the preceding hypothesis that $D(PQ)=D^*(P^*Q^*)$; in fact, there is induced in space a quadratic differential form, the same form in γ, ϕ, θ and in $\gamma^*, \phi^*, \theta^*$. If x^i are rectangular Cartesians, there exist simple geometrical transformations $x^i=f^i(\gamma, \phi, \theta)$, $x^{*i}=f^{*i}(\gamma^*, \phi^*, \theta^*)$, and we are led to the metric $ds^2=g_{ij}dx^idx^j=g_{ij}dx^{*i}dx^{*j}$, which in general is non-Euclidean. Considerations of symmetry reduce the metric to $ds^2=M^2(\gamma)(\sigma^2d\gamma^2+d\phi^2+\cos^2\phi d\theta^2)$, where σ is a constant, and facts of observation lead the author to conclude that it is of negative constant curvature, so that the geometry is of hyperbolic type. *J. L. Synge* (Pittsburgh, Pa.).

Esnault-Pelterie, Robert. Remarques sur une formule usuelle. C. R. Acad. Sci. Paris **224**, 1404–1407 (1947).

Esnault-Pelterie, Robert. Remarques sur une formule usuelle. C. R. Acad. Sci. Paris **224**, 1462–1464 (1947).

Esnault-Pelterie, Robert. Sur la répartition des produits d'un ensemble de fécondations avec croisements libres. C. R. Acad. Sci. Paris **224**, 1796–1799 (1947).

Esnault-Pelterie, Robert. Sur la répartition des produits d'un ensemble de fécondations avec croisements libres. C. R. Acad. Sci. Paris **225**, 14–16 (1947).

Consider a population producing two species of gametes whose proportions in any generation are $a:b$, with $a+b=1$. Let $a-b=2\tau-1$. The author compares several modes of variation of τ and the consequent changes in the proportion $a:b$. Most interesting are the cases of selective mating, where the proportions x and y of homozygotes and h of hybrids in the succeeding generations can be represented by

$$x:h:y=(1+\alpha)\tau^2:2(1+\gamma)\tau(1-\tau):(1+\beta)(1-\tau)^2,$$

α, β and γ measuring the force of the selection. The difference equations are replaced by differential equations and integrations effected for each of 5 cases. Sufficiently representative is the case $0=\beta=\gamma\neq\alpha$ where (in units of a generation time)

$$t=\alpha^{-1}[\log(\tau/\tau_0)+(1+\alpha)\log\{(1-\tau_0)/(1-\tau)\}+(\tau_0^{-1}-\tau^{-1})].$$

Taking $\alpha=\tau_0=0.001$, the author finds that x remains negligible (<0.03) for a million generations, but thereafter increases rapidly, and in 10,000 more generations constitutes virtually the entire population. *A. S. Householder.*

Andreoli, Giulio. Schema statistico di evoluzione e di selezione in una collettività a monoibridismo mendeliano. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 260–280 (1940).

Elementary computations of the development of a Mendelian population under certain simple modes of selection. No references are made to existing literature. *W. Feller.*

Mathematical Economics

***von Neumann, John, and Morgenstern, Oskar. Theory of Games and Economic Behavior.** 2d ed. Princeton University Press, Princeton, N. J., 1947. xviii+641 pp. $10.00.

[The first edition appeared in 1944; cf. these Rev. **6**, 235.] The second edition differs only slightly from the first. The changes consist essentially of eliminating the misprints discovered in the first edition and of adding an appendix which contains a derivation of the numerical character of utility from the axioms formulated in section 3. As the authors remark, the axioms in their present form do not allow for the possibility of a specific utility or disutility of gambling. To permit such a possibility, it would be necessary to modify some of the axioms. It seems likely that the really critical axiom that would have to be changed is the axiom (3:C:b) which expresses the combination rule for multiple chance alternatives. The mathematical difficulties in formulating a system which allows for a utility or disutility of gambling are likely to be considerable and, as is pointed out by the authors, the current method of using indifference curves would not diminish these difficulties. *A. Wald* (New York, N. Y.).

Palomba, Giuseppe. Elementi matematici per l'economia corporativa. (Il problema di "minimo" posto dalle corporazioni). Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 303–309 (1940).

TOPOLOGY

***Brödel, W. Beweis des Vierfarbensatzes.** Ber. Math.-Tagung Tübingen 1946, pp. 47–49 (1947).

The author announces a solution of the four color problem. The problem is first reduced to the construction of a closed Riemann surface over the sphere with certain properties, and then to the determination of a subgroup of a known group of linear transformations. From the brief indications of the author the reviewer is unable to see how this can essentially alter the combinatorial character of the problem. The mathematical world must remain in suspense until details of the proof are published. *L. Ahlfors.*

De Backer, S. M. Contribution au problème des quatre couleurs. Emploi des congruences module 5. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 441–453 (1947).

The author applies congruences modulo 5 to nets dual to regular maps. He thus proves that a minimal irreducible map cannot contain a pentagon touching two other non-consecutive polygons each of which is a pentagon or a hexagon. *P. Franklin* (Cambridge, Mass.).

Dubois-Violette, Mme. Pierre-Louis. Sur une extension topologique d'une formule de Poincaré. C. R. Acad. Sci. Paris **224**, 625–627 (1947).

This paper is concerned with a net of curves on a sphere and their singularities. Such a net can be obtained, by stereographic projection, from the solutions of differential equations of first order and first degree. Poincaré has shown that if four types of singularities of simple type arise there is a relation between them. The author is concerned with an extension of this formula by topological means. *M. R. Hestenes* (Los Angeles, Calif.).

Dubois-Violette, Mme. Pierre-Louis. Sur une signification topologique et une généralisation d'une formule de Bendixson. C. R. Acad. Sci. Paris **224**, 706–708 (1947).

The present note is concerned with the classification of singularities of solutions of a differential equation of first order and first degree. It is pointed out that by a geometrical argument it can be shown that a singularity is equiva-

lent to singularities of the elementary types previously discussed by the author [see the preceding review].

M. R. Hestenes (Los Angeles, Calif.).

Dubois-Violette, Mme. Pierre-Louis. Sur les réseaux de courbes sans point singulier tracés sur une surface de genre 1. C. R. Acad. Sci. Paris **224**, 782–784 (1947).

Consider a net of curves covering, without singularities, a surface of genus 1. In terms of such a net the author defines an index of a simply closed curve on the surface and studies the properties of this index. *M. R. Hestenes.*

Lyusternik, L. A. Topology and the calculus of variations. Uspehi Matem. Nauk (N.S.) **1(11)**, no. 1, 30–56 (1946). (Russian)

An expository article, surveying and assembling results obtained by applying topological methods to the study of critical points of functions and functionals; except for some references to G. D. Birkhoff and the earlier papers of M. Morse, only the Russian research is discussed. Functions on manifolds are discussed first; Morse's use of Betti numbers is mentioned, then the results obtained by use of "category." For example, if F is analytic on a torus, Morse's theorems yield the corollary that the sum of the multiplicities of the critical points of F is at least 4; use of category leads to the conclusion that there are at least three geometrically distinct critical points. The use of "length" is mentioned [cf. L. Elsholz, Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 565–571 (1939); these Rev. **1**, 317].

In applications to functionals, such as the study of geodesics on a manifold, analytic difficulties are intentionally ignored. The critical values of a functional J on a space R are, essentially, the values at which the topology of the set $R[J<c]$ changes. The methods rely greatly on Pontrjagin's "removing theorem" [cf. P. Alexandroff, Trans. Amer. Math. Soc. **54**, 286–339 (1943); these Rev. **5**, 48]. The theorems yield, as instances, known theorems on the existence of three closed geodesics on surface homologous to spheres, and infinitely many geodesics joining two fixed points of such surfaces. References are to several papers by Lyusternik already reviewed.

The paper concludes with a discussion of a sharpened form of Poincaré's method of continuation.

E. J. McShane (Charlottesville, Va.).

Zaremba, S. K. Une généralisation de l'indice de Kronecker relative aux familles de courbes dans des variétés à trois dimensions. Revista Ci., Lima **48**, 357–401 (1946).

Hamburger [Math. Z. **19**, 50–66 (1923)] has defined an index for each isolated singular point of a regular family of curves on a two-dimensional manifold. In the present work an analogous definition is made for the case of a family of curves in a three-dimensional manifold. Here, however, it is essential that the curves be oriented. It is shown that, if the manifold is compact and if there are only a finite number of singular points, then the sum of their indices is zero. *E. G. Begle* (New Haven, Conn.).

Borsuk, Karol. Contribution au problème de l'unicité de la décomposition en produits cartésiens. C. R. Soc. Sci. Varsovie **33–38**, 1–4 (1946).

Previously [Fund. Math. **31**, 137–148 (1938); **33**, 273–298 (1945); these Rev. **8**, 48] the author has shown that under certain conditions decomposition of a topological space into a Cartesian product of prime divisors is unique. The present contribution sketches two further results of this nature. (1) In any decomposition of a polytope (locally polyhedral space) E into a Cartesian product $A_1 \times A_2 \times \cdots \times A_n \times M$ of a manifold (connected locally Euclidean space) M and a finite collection of 1-dimensional continua A_i, none of which are simple closed curves, the topological type of the divisors A_i and the dimension and homology characters of the divisor M are uniquely determined by E. (2) No polyhedron can have more than one decomposition into a product of prime divisors each of which is either of dimension not exceeding 1 or is a union of a finite collection of nonintersecting compact closed surfaces, simple closed curves and isolated points.

R. H. Fox (Princeton, N. J.).

Paintandre, Roger. Sur une classe d'espaces topologiques. C. R. Acad. Sci. Paris **224**, 1806–1808 (1947).

This paper contains preliminary results about topological spaces in which each point has a linearly ordered basis of neighborhoods. *R. Arens* (Los Angeles, Calif.).

Paintandre, Roger. Sur l'extension de la théorie des fonctions de Baire à une classe d'espaces topologiques non métriques. C. R. Acad. Sci. Paris **225**, 26–28 (1947).

It is shown that Baire's theorem on points of continuity of functions of a given class fails for some completely regular spaces. A satisfactory generalization of Baire's theorem is then established for functions whose domain and range possess certain properties considered in the note reviewed above. *R. Arens* (Los Angeles, Calif.).

Paintandre, Roger. Sur la construction de sphères généralisées dans une classe d'espaces topologiques. C. R. Acad. Sci. Paris **225**, 92–94 (1947).

The author establishes the equivalence of two types of spaces defined by him in an earlier note [cf. the second preceding review]. *R. Arens* (Los Angeles, Calif.).

Stone, M. H. Remarks on metrizability. Bull. Amer. Math. Soc. **53**, 289–291 (1947).

L'auteur démontre de façon nouvelle le théorème connu d'après lequel tout espace régulier parfaitement séparable est métrisable. Pour cela il commence par établir que: si R est un espace régulier parfaitement séparable, alors R est homéomorphe à l'espace obtenu en introduisant une topologie dite "topologie faible" dans $\mathfrak{X}$, où $\mathfrak{X}$ est une famille dite "continue" de sous-ensembles fermés, non vides et disjoints d'un espace métrique compact S qui peut être pris comme un sous-ensemble fermé du discontinu cantorien. L'auteur démontre ensuite que l'espace obtenu en introduisant la topologie faible dans une famille d'ensembles jouissant des propriétés ci-dessus de $\mathfrak{X}$ (et même dans une famille un peu plus générale) est métrisable; d'où résulte le théorème envisagé.

La famille $\mathfrak{X}$ est dite "continue" (notion due sous un autre nom à R. L. Moore) si pour tout $X_0 \in \mathfrak{X}$ et pour tout voisinage ouvert G de X_0 il existe un voisinage ouvert G_0 de X_0 tel que chaque $X \in \mathfrak{X}$ ayant un point commun avec G_0 soit contenu dans G (on appelle voisinage ouvert d'un sous-ensemble X_0 de S tout sous-ensemble ouvert de S contenant X_0). La "topologie faible" dans $\mathfrak{X}$ est obtenue en prenant comme famille de voisinages de chaque élément $X_0 \in \mathfrak{X}$ la famille de tous les ensembles de la forme $U(X_0, G)$ = ensemble de tous les $X \in \mathfrak{X}$ tels que $X \subset G$, où G est un voisinage ouvert arbitraire de X_0. *A. Appert* (Saumur).

Alexandroff, P. On the dimension of normal spaces. Proc. Roy. Soc. London. Ser. A. **189**, 11–39 (1947).

Theorems connecting the Lebesgue covering dimension with homology properties and mappings in spheres, known for separable metric spaces (and largely also due to Alexandroff), are generalized to normal spaces. The proofs are developed by exploiting the spectrum of various types of coverings to achieve the results for bicompact spaces, and using the fact that a normal space can be extended to a bicompact space of the same covering dimension. [Strong similarities exist with the work of Hemmingsen [Duke Math. J. **13**, 495–504 (1946); these Rev. **8**, 334], and Dowker [Amer. J. Math. **69**, 200–242 (1947); these Rev. **8**, 594], for example the result that if R is a normal space then $\dim R \leqq n$ if and only if every mapping of every closed subset of R into S^n can be extended over R; this is Alexandroff's theorem 4.50, p. 26; Hemmingsen's theorem 3.1, p. 498; Dowker's corollary 36, p. 212.] "Abstract" Cantor manifolds are defined and proved to exist in any n-dimensional normal space. A number of open problems in general dimension theory are indicated. *H. Wallman.*

Eilenberg, Samuel. Homology of spaces with operators. I. Trans. Amer. Math. Soc. **61**, 378–417; errata, **62**, 548 (1947).

Let C be an abstract complex, W a group of automorphisms of C. For an Abelian coefficient group G, in which W acts as group of operators, "equivariant" cohomology groups $H_w^n(C, G)$ are defined for all dimensions $n \geqq 0$; they are based upon cochains f^n such that $f^n(wc_n) = w \cdot f^n(c_n)$ for all n-cells c_n and all $w \in W$. If W acts trivially on C and G, the $H_w^n(C, G)$ are the ordinary cohomology groups of C. Let X be a topological space, W a group of homeomorphisms of X onto itself; the singular complex $C(X)$ of X admits W as group of automorphisms, and the groups $H_w^n(C(X), G)$ are called the equivariant cohomology groups $H_w^n(X, G)$ of X. They are simultaneous topological invariants of X and W; if X is a polyhedron and if W acts simplicially on X, they may be computed from the simplicial scheme of X. When X is a regular covering of a space Y, and W the corresponding group of covering transformations, the $H_w^n(X, G)$ are isomorphic with the cohomology groups of Y with suitable local coefficients in the sense of Steenrod [Ann. of Math. (2) **44**, 610–627 (1943); these Rev. **5**, 104].

Let $H^n(W, G)$ denote the algebraic cohomology groups of an abstract group W, as introduced by Eilenberg and MacLane [Ann. of Math. (2) **48**, 51–78 (1947); these Rev. **8**, 367] and the reviewer [Comment. Math. Helv. **18**, 232–282 (1946); these Rev. **8**, 166]. The following theorem is proved. If the homeomorphism group W acts in the space X without fixed points, and if the ordinary homology groups of X vanish in dimensions less than q, then $H_w^n(X, G) \cong H^n(W, G)$, $n < q$; this result extends partially to dimensions $n = q$ and $n = q+1$. This theorem yields the generalization to local coefficients of the known relations between the fundamental group of an aspherical space and its cohomology groups (established recently for ordinary coefficients by Hopf, Eilenberg and MacLane, Freudenthal and the reviewer). The complete discussion is carried out for both homology and cohomology, in connection with exact homomorphism sequences, and sketched for the product theory.

B. Eckmann (Lausanne).

Eilenberg, Samuel. Singular homology in differentiable manifolds. Ann. of Math. (2) **48**, 670–681 (1947).

This is a continuation of the author's paper in the same Ann. (2) **45**, 407–447 (1944); these Rev. **6**, 96. The main objective is to show that in studying the singular homology theory of a manifold M of class C^k one may confine attention to the singular simplexes of class C^k. In the earlier work the author defined a singular simplex as a mapping $T: s \to M$ of a nondegenerate Euclidean simplex s with ordered vertices; equivalence between simplexes was defined in such a way as to eliminate certain objections in the older singular theory. Now T is said to be of class C^k if it is a mapping of class C^k. The totality of such simplexes is a subcomplex $S^k(M)$ of the full singular complex $S(M)$. The main result is that $S(M)$ and $S^k(M)$ have the same homology characters. This follows from the proposition that the identity mapping $\epsilon: S^k(M) \to S(M)$ is a chain-equivalence, which means that there exists a chain-transformation $\bar{\epsilon}: S(M) \to S^k(M)$ such that $\epsilon\bar{\epsilon}$ and $\bar{\epsilon}\epsilon$ are chain-homotopic to the identity automorphisms in $S(M)$ and $S^k(M)$. The proof requires a number of lemmas about differentiable functions. There is an appendix about complexes with operators.

P. A. Smith (New York, N. Y.).

Kelley, J. L., and Pitcher, Everett. Exact homomorphism sequences in homology theory. Ann. of Math. (2) **48**, 682–709 (1947).

A sequence $\{G_i\}$ of groups and of homomorphisms $\varphi_i: G_i \to G_{i-1}$ is called exact if the kernel of φ_i coincides with the image of φ_{i+1}. The occurrence of such sequences in homology theory was first observed by Hurewicz [Bull. Amer. Math. Soc. **47**, 562–563 (1941)] and has since developed into a fundamental tool in homology and homotopy. The authors systematically exhibit exact sequences that occur in connection with homology and cohomology theory of abstract complexes, Čech theory, singular theory and manifolds.

As an application the case of a complex covered by a finite family of subcomplexes is considered. Relations are obtained between the homology groups of the complex, the covering subcomplexes and their various intersections, and of the nerve of the covering. These results bear some similarity to recently announced definitions by Leray [C. R. Acad. Sci. Paris **222**, 1366–1368, 1419–1422 (1946); these Rev. **8**, 49]. In the final section an abstract derivation of the critical level relations of Morse is obtained.

S. Eilenberg (New York, N. Y.).

Leray, Jean. Une définition géométrique de l'anneau de cohomologie d'une multiplicité. Comment. Math. Helv. **20**, 177–180 (1947).

A very brief, intuitive and unrigorous description of the cohomology ring of a manifold. *S. Eilenberg.*

Whitehead, George W. Correction to my paper "On families of continuous vector fields over spheres." Ann. of Math. (2) **48**, 782–783 (1947).

The proof of lemma 1 in the paper cited in the title [same Ann. (2) **47**, 779–785 (1946); these Rev. **8**, 167] contains a mistake. A corrected proof is given in this note.

H. Samelson (Ann Arbor, Mich.).

Hirsch, Guy. Un théorème sur les transformations des sphères. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 394–399 (1947).

Theorem: if an n-sphere is mapped continuously into itself, (1) some pair of antipodal points is mapped into a single point and/or (2) some pair of antipodal points is mapped into a pair of antipodal points. For, if (1) fails,

the mapping has odd degree in consequence of Borsuk's theorem; and, if (2) fails, the mapping has even degree [actually degree zero when n is even]. The author gives some corollaries and raises questions concerning generalizations. *A. W. Tucker* (Princeton, N. J.).

Hall, D. W., and Youngs, J. W. T. Comments on the cores of certain classes of spaces. Ann. of Math. (2) **48**, 710–716 (1947).

Suppose there is given a fixed class τ of transformations $f(X)=Y$. Let $\mathfrak{X}$ be the class of all spaces X, and $\mathfrak{Y}$ the class of all image spaces Y, for $f\in\tau$. The following assumptions are made: (1) $\mathfrak{X}=\mathfrak{Y}$, (2) for each $X\in\mathfrak{X}$, τ contains the identity transformation, and (3) if $f_1(X)=Y$ and $f_2(Y)=Z$, with $f_1\in\tau$ and $f_2\in\tau$, then $f_2f_1(X)=Z$ with f_2f_1 in τ. Then the class $\mathfrak{X}$ of spaces is topologized by the following closure definition. If $\mathfrak{A}$ is a subclass of $\mathfrak{X}$ then $\overline{\mathfrak{A}}$ is the collection of all spaces $f(A)$ for $A\in\mathfrak{A}$, $f\in\tau$. These theorems hold: (1) $\overline{\cup\mathfrak{A}}=\cup\overline{\mathfrak{A}}$, (2) $\overline{\mathfrak{A}}\supset\mathfrak{A}$, and (3) $\overline{\overline{\mathfrak{A}}}=\overline{\mathfrak{A}}$. For $\mathfrak{A}\subset\mathfrak{X}$, the core of $\mathfrak{A}$, written $C(\mathfrak{A})$, is defined as the union of all closed subclasses of $\mathfrak{A}$. In view of theorem (1), $C(\mathfrak{A})$ is the largest closed subset of $\mathfrak{A}$. Theorem 4. If $\mathfrak{A}$ is determined as the set of all spaces $A\in\mathfrak{X}$ having a given property π, then $C(\mathfrak{A})$ is the set of all spaces having the property persistently π; i.e., $A\in C(\mathfrak{A})$ provided, for every $T\in\tau$, $T(A)$ has property π.

In the applications, τ is taken as the family of monotone mappings, and $\mathfrak{X}$ the class of Peano spaces. Let $\mathfrak{U}$, $\mathfrak{C}$, $\mathfrak{B}$, $\mathfrak{J}$, $\mathfrak{R}$ be, respectively, the class of all Peano spaces which are unicoherent, cyclic, bicyclic, Jordan curves or single points, retractable onto any one of their subarcs. The following are some of the theorems: $C(\mathfrak{R})=\mathfrak{U}$, $C(\mathfrak{B})=\mathfrak{U}$, $C(\mathfrak{C})=\mathfrak{J}$. *J. H. Roberts* (Durham, N. C.).

***de Kerekjarto, Bela. Topologia dei gruppi di trasformazioni delle superficie.** Atti Convegno Mat. Roma 1942, pp. 35–47 (1945).

GEOMETRY

Melmore, Sidney. Partitioning of space into enantiomorphous polyhedra. Philos. Mag. (7) **37**, 574–576 (1946).

Nachdem erstmals W. Barlow [Philos. Mag. (6) **46**, 930–956 (1923)] ein enantiomorphes Polyeder angegeben hat, mit dem man den ganzen Raum ausfüllen kann, wird in vorliegender Arbeit durch eine einfache geometrische Konstruktion ein weiteres solche Polyeder gefunden. Es wird aus einem Rhombendodekaeder erhalten, indem man zwei passend gewählte Seitenflächen wachsen lässt. *J. J. Burckhardt* (Zürich).

Melmore, Sidney. Densest packing of equal spheres. Nature **159**, 817 (1947).

The densest packing of equal circles in the plane is unique: each circle is surrounded by six others. Spheres on these circles as equators form a "hexagonal layer." It is obvious (though nobody has yet succeeded in proving it) that every densest packing of equal spheres is made up of such hexagonal layers, which we may conveniently take to be horizontal. Each sphere of one layer fits in between three of the previous layer; but only half these interstitial positions can be occupied, so there is a choice of two possibilities for each successive layer. [See W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, pp. 149–150; these Rev. **8**, 440.] Let us denote each layer by a letter A or B or C, writing ABA if the third layer is vertically above the first, but $ABCA$ if it is the fourth that is above the first. The arrangement of twelve spheres around one of the B spheres is different in the two cases: say T in the former case, R in the latter (and then the C spheres are likewise of type R, having B on one side and A on the other). Let a packing be called homogeneous if it has a symmetry group that is transitive on the T spheres and on the R spheres. There are six possibilities: every T layer, and likewise every R layer, must be sandwiched between either two T's or two R's or one of each. Using dots to indicate periodicity, as in a recurring decimal, we find the six homogeneous packings to be

$$\dot{T}=A\dot{B},\quad \dot{R}=A\dot{B}C,\quad T\dot{R}=ABC\dot{B},\quad \dot{R}T\dot{R}=ABACB\dot{C},$$
$$T\dot{R}T=ABCBCACAB\dot{A},\quad T\dot{R}R\dot{T}=ABCACABCBCAB\dot{A}.$$

The first two are the familiar hexagonal and cubic packings; the rest are "homogeneous mixtures" of them. *H. S. M. Coxeter* (Toronto, Ont.).

***Maier, W. Pyramidenmessung durch Dilogarithmen.** Ber. Math.-Tagung Tübingen 1946, pp. 104–106 (1947).

This paper is concerned with Abel's remarkable identity

$$\sum_{\rho=1}^{5}\int_0^{t_\rho}\left(\frac{\log(1-z)}{z}+\frac{\log z}{1-z}\right)dz=-\pi^2,$$

where $t_\rho=t_{\rho+1}=1-t_{\rho-2}t_{\rho+2}$. [See Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938, p. 265.] The author seeks to relate it, by a method that is not made clear, to the Pentagramma Mirificum of Napier and Gauss, and to a special case of Lobatschefsky's formula for the volume of a quadrirectangular tetrahedron. [Cf. Bloch and Guillaumin, C. R. Acad. Sci. Paris **224**, 1690–1692 (1947); these Rev. **8**, 596.] *H. S. M. Coxeter* (Toronto, Ont.).

Dixon, A. L. On a formula connecting one measure of distance with another. Quart. J. Math., Oxford Ser. **18**, 128 (1947).

When the familiar formula for the cross ratio of four lines of a pencil is expressed in terms of points on the elliptic line, it becomes $\{AB, PQ\}=\sin AP \sin BQ/\sin BP \sin AQ$. Hence the cross ratio of four points on the hyperbolic line is $\{AB, PQ\}=\sinh AP \sinh BQ/\sinh BP \sinh AQ$. If we wish to change the hyperbolic metric so as to use AB as a new pair of absolute points, the new distance PQ will be $\frac{1}{2}\log\{AB, PQ\}$, where the cross ratio is given, in terms of the original metric, by the above formula. *H. S. M. Coxeter* (Toronto, Ont.).

Haantjes, J. Distance geometry. Curvature in abstract metric spaces. Nederl. Akad. Wetensch., Proc. **50**, 496–508 = Indagationes Math. **9**, 302–314 (1947).

Let B be a rectifiable arc in a metric space M. For each pair of points q, r of B, $d=d(qr)$ denotes the distance of q and r, $l=l(qr)$ the length of the subarc qr of B, $K(q, r)$ the expression $\{4!(l-d)l^{-3}\}^{\frac{1}{2}}$. The curvature of B at a point p of B is defined as the limit $K=K(p)$, if it exists, of $K(p, q)$, when q and r converge independently to p. The Menger and the Alt curvatures are denoted by K_M and K_A. The following theorems are proved. (5) If M is a Riemannian space and the classical curvature (in the sense of Schouten) $C=C(p)$ exists, $K(p)$ also exists and is equal to C. (6) The curvature K is a continuous function of p wherever it exists.

(7) If both K_A (or K_M) and K exist, they are equal. (8) If $K_A(p)$ exists, it equals $\lim K(p, q)$ when $q \to p$. (9) The existence of K_M implies the existence of K and the two are equal. (10) An arc in a Euclidean space with a curvature K at a point p has a tangent in a neighbourhood of this point. (11) For arcs in a Euclidean plane the notion of curvature K is equivalent to the notion of Menger curvature.

An example is given of an arc possessing a curvature K at a point without having a Menger curvature or even an Alt curvature at this point. An interesting example of analytical computation in local distance geometry is the proof of theorem 8, where $l(qr)$ and $d(qr)$ are compared when $K_A(p)$ exists and q and r approach p. *C. Pauc* (Marseille).

Thébault, Victor. Tetrahedrons having a common face. Amer. Math. Monthly **54**, 395–398 (1947).

Trost, Ernst. Eine kennzeichnende Eigenschaft des Kreises. Elemente der Math. **2**, 76–80 (1947).

For a plane curve C let D_α be a triangle SA_1A_2, where SA_1 is a tangent of C at A_1 and α is the angle at S. If the Δ_α are isosceles ($SA_1 = SA_2$) for every α, then C is a circle. A general expression is derived for all curves for which a particular angle α exists such that the Δ_α are isosceles, and various examples are discussed. The circle can be characterized by the condition that an α exists for which the Δ_α are congruent. *H. Busemann* (Los Angeles, Calif.).

Belgodère, Paul. Géométries élémentaires et nombres complexes. Gaz. Mat., Lisboa, nos. **30, 31, 32**, 40 pp. (1947).

The author uses complex and several types of hypercomplex numbers to discuss geometries associated with various groups of transformations. *E. Lukacs.*

Bompiani, E. Sulle jacobiane di una corrispondenza puntuale fra piani. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 22–26 (1947).

Si O est un point simple de la jacobienne du plan π, une direction unique E_1, stationnaire, correspond à toutes les directions E_1' issues de O dans π'. Le cas où cet E_1 est distinct de la tangente en O à la jacobienne a déjà été étudié [Bompiani, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **14**, 11–21 (1943); ces Rev. **8**, 219]. Les éléments de contact du deuxième ordre, exceptionnels, qui donnent lieu, dans la transformation, à des éléments de rebroussement ou d'inflexion d'espèce supérieure, sont ici étudiés dans le cas où cet E_1 stationnaire est confondu avec la tangente en O à la jacobienne. Il existe alors, en général, une forme canonique de la transformation au voisinage de (O, O'),

$$x' = y + x^2 + x^3 + 3a_{21}x^2y + a_{03}y^3 + [4],$$
$$y' = xy + Ix^3 + b_{03}y^3 + [4],$$

les repères des coordonnées projectives étant univoquement déterminés, et les coefficients étant donc des invariants. *P. Belgodère* (Paris).

Villa, M. Sulle trasformazioni puntuali in una coppia a jacobiano nullo nel caso cremoniano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 136–140 (1947).

En liaison avec la note précédente de E. Bompiani, il est montré que l'appartenance à la jacobienne de l'E_1 stationnaire est la condition nécessaire et suffisante pour que la transformation donnée soit approchable, jusqu'au voisinage du 2e ordre de (O, O'), par une transformation birationnelle. L'étude des transformations cubiques osculatrices fournit une interprétation géométrique des repères projectifs canoniques, et des invariants, jusqu'au voisinage du 3e ordre. *P. Belgodère* (Paris).

Villa, Mario. Sulle trasformazioni puntuali in una coppia a jacobiano nullo. Boll. Un. Mat. Ital. (3) **2**, 3–8 (1947).

Buzano, Piero. Studio di alcuni sistemi ∞^1 di omografie piane. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **80**, 65–71 (1945).

Les seuls systèmes ∞^1 d'homographies planes, contenant l'identité, et tels que si P est un point quelconque du plan, et Q un point de sa trajectoire par le système considéré, les tangentes en ces points à cette trajectoire se correspondent dans le même homographie qui fait passer de P à Q, sont: (a) les groupes à un paramètre d'homographies planes, (b) les systèmes d'homologies de même centre. *P. Belgodère* (Paris).

Turri, Tullio. Sulle anticorrelazioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 274–297 (1940).

An anticorrelation in n-space is defined by $\sigma u_j' = \sum a_{jk}\bar{x}_k$, $j, k = 0, \cdots, n$, where (x) is a point, (u) is a hyperplane, and $\bar{x}$ is the complex conjugate of x. If A is the matrix of this anticorrelation, then $(A')^{-1}\bar{A}$ is the matrix of the corresponding homology. The author determines the types of indecomposable anticorrelations and calculates the number of projectively distinct anticorrelations having the same homology. *C. C. MacDuffee* (Madison, Wis.).

Turri, Tullio. Semplificazione di dimostrazione nella nota: "La continuità delle trasformazioni conservanti i gruppi armonici." Rend. Sem. Fac. Sci. Univ. Cagliari **15** (1945), 216–222 (1947).

The note in question appeared in the same Rend. **15**, 90–97 (1946); these Rev. **8**, 483.

***Schmidt, Hermann. Eine Verallgemeinerung der projektiven Inversion.** Ber. Math.-Tagung Tübingen 1946, pp. 129–130 (1947).

This is an abstract of the author's work on (2, 2) correspondences in n-dimensional Euclidean space. The correspondence is defined by means of a fixed quadratic hyperspace V^2 and fixed point O not on V^2. Let x be an arbitrary point and denote by p and q the two points of intersection of Ox with the V^2. A point y corresponds to x if the cross ratio $(p\,q, x\,y) = \alpha^{\pm 1}$. *E. Lukacs* (Cincinnati, Ohio).

Rodeja F., E. G. Aire de l'ellipse déterminée par cinq points. Comment. Math. Helv. **20**, 172–176 (1947).

The solution is based upon the expression for the radius of the circle circumscribed to a triangle as the product of the three sides divided by four times the area. Five points on the circumference of a circle give rise to ten different triangles which give the same value for the radius. An expression is obtained for the area of the circle in terms of the areas of the ten triangles. This relation is unchanged by a projection and therefore applies to an ellipse. The final expression is not symmetrical in all the ten triangles. A number of analogous expressions could be formed by a cyclical change. The problem was suggested by the possible application to the determination of the apparent ellipse of visual double stars. The practical value for this problem appears to be limited to exceptional cases. *D. Brouwer.*

Bradley, A. Day. Great circle trigonometry. Scripta Math. 13, 64–70 (1947).

Convex Domains, Extremal Problems

***Combes, Bernard. Le Plus Petit Corps Convexe Contenant un Arc de Courbe dans l'Espace à n Dimensions.** Thesis, University of Paris, 1946. 29 pp.

A sequence of functions (1) $X_i=\varphi_i(x)$, $a\leqq x\leqq b$, may be thought of as defining a curve Γ in space of infinitely many dimensions. Let Γ_n be the curve in n-dimensional space S_n defined by (1) with $i=1, \cdots, n$. Let Ω_n be the convex hull of Γ_n. Let δ_n for a given direction D in S_n denote the maximum of the lengths of the chords of Ω_n which are parallel to D. The points of Ω_n are representable in the form $M_i=\int_a^b \varphi_i(x)dF(x)$ with a distribution function F. If D is the X_p-axis, then δ_n measures how accurately M_p is determined by $M_1, \cdots, M_{p-1}, M_{p+1}, \cdots, M_n$. The convex hull of Γ is said to be "infinitely thin" if for each n we can find a direction D_n such that the corresponding δ_n converge to zero. If $D_n=D$ independent of n, the convex hull of Γ is called infinitely thin in the direction D. If the convex hull of Γ is infinitely thin in the X_p-direction, M_p is uniquely determined by all the other M_i.

The author applies these notions to the special curve defined by $X_i=x^i$ ($i=1, 2, \cdots$). In this case the M_i are the ordinary moments of the mass distribution defined by F. He takes for D the X_p-axis and derives estimates for the corresponding δ_n as functions of a, b, n. He finds, e.g., that the convex hull is infinitely thin in the direction of each coordinate axis if $0\leqq b-a\leqq 4$, $b-5a\leqq 0$. *F. John.*

***Süss, W. Kennzeichnende Eigenschaften der Kugel als Folgerung eines Brouwerschen Fixpunktsatzes.** Comment. Math. Helv. 20, 61–64 (1947).

By means of Brouwer's fixed point theorem for the sphere the author proves the following theorem. Let E be a convex body in three-space with the following property: to every direction ξ corresponds a plane normal to ξ, depending continuously on ξ, and intersecting E in a region $B(\xi)$ such that all these regions $B(\xi)$ are congruent. Then the regions $B(\xi)$ are circles. From this several known characterizations of the sphere among the convex bodies are easily deduced, e.g.: if all plane intersections through a fixed point are congruent, the body is a sphere. Generalizations where the congruence group is replaced by other groups of geometrical transformations are mentioned. *W. Fenchel.*

Kubota, Tadahiko. Some inequalities concerning ovals and ovaloids. Tôhoku Math. J. 49, 213–219 (1943).

The author obtains an inequality (*) $O\geqq 4\Delta^2/\sqrt{3}$ between the surface area O and the smallest breadth Δ of a convex body in Euclidean 3-space as an immediate consequence of the well-known Cauchy formula for the surface area and an inequality concerning ovals due to Pál. In contrast to the latter, (*) is not sharp. Furthermore, an inequality due to the author [Math. Z. 20, 264–266 (1924)] between area, length and diameter of a plane oval is proved in a new way. *W. Fenchel* (Copenhagen).

***Blaschke, Wilhelm. Sugli ombelichi di un ovaloide.** Atti Convegno Mat. Roma 1942, pp. 201–208 (1945).

Algebraic Geometry

***Gröbner, Wolfgang. L'algebra moderna e la geometria algebrica.** Atti Convegno Mat. Roma 1942, pp. 215–222 (1945).

Barbilian, D. Axiomatische Begründung des Abelschen Theorems im Grossen. Disquisit. Math. Phys. 1, 5–22 (1940).

An additively written Abelian group G is called uniformly bound if the equation $nX=B$, for any $n>1$, has at least two but only a finite number of solutions. It is proved that in such group the solutions to $nX=0$ form a finite group with r equal torsion coefficients ($=n$), where r is an invariant of G. (G is then said to have r periods.)

Let F be an irreducible plane curve having at the origin an s-fold point S with s distinct tangents. Two plane curves, or two polynomials, are said to define the same cut on F at S if one is mod F a unit power-series times the other. The cut is called a place if it has a representative having at S an $(s-1)$-fold point. Let O_0 be a fixed place at S having no tangent in common with F at S, O_1, O_2 two other places at S. It is possible to find a place O_{12} such that $O_1O_2=O_0O_{12}$, where O_1O_2 is the cut defined by the product of polynomials defining O_1, O_2. If O_{12} is defined as the sum of O_1 and O_2, the places at S form a uniformly bound group with $s-1$ periods, called the accessory group of S. Let g_0 be a fixed set of p points on F, where p is the genus of F; g_1, g_2 two other sets of p points each. Then it is possible to find a set g such that $g_1+g_2=g+g$, where "$=$" means coresidual. If g is defined as the sum of g_1 and g_2, then $\{g\}$ becomes a uniformly bound group with $2p$ periods, called the kernel group. The definition of this group involves, with the adjoints, all the singularities of F; for each subset of the singularities, however, a group similar to the kernel group can be defined and is called the essential group for this subset. An essential group is shown to be the direct product of the corresponding accessory groups and the kernel group. [The inequality in formula (9′) does not appear to have been established.] *A. Seidenberg* (Berkeley, Calif.).

Châtelet, François. Les courbes de genre 1 dans un champ de Galois. C. R. Acad. Sci. Paris 224, 1616–1618 (1947).

The Galois methods of the author's earlier paper [Ann. Univ. Lyon. Sect. A. (3) 9, 40–49 (1946); these Rev. 8, 565] are used to give a simple proof of the theorem that if a curve defined in a Galois field R is irreducible and of genus one in every finite extension of R, then it contains a point of R. Earlier proofs of this result required a theorem of F. K. Schmidt [Math. Z. 33, 1–32 (1931)] demonstrated by transcendental methods involving zeta functions. *D. B. Scott* (Aberdeen).

Zappa, Guido. Sul gruppo fondamentale delle curve di diramazione delle superficie algebriche suscettibili di spezzarsi in sistemi di piani. Ann. Mat. Pura Appl. (4) 24, 139–151 (1945).

The author determines the fundamental group of the branch curve $f(x, y)=0$ of an algebraic surface $F(x, y, z)=0$, of order m, under the assumption that F can be made to degenerate into m distinct planes, no three coaxial. For any pair (i, j) of these planes such that their common line is not part of the limit of the double curve of F, a suitable element g_{ij} of the fundamental group is introduced. It is then shown that the g_{ij} generate this group and that the following is a

complete set of generating relations: $g_{ij}g_{iu}g_{ij}=g_{iu}g_{ij}g_{iu}$ $(j\neq u)$, $g_{ij}g_{uv}=g_{uv}g_{ij}$ $(i,j\neq u,v)$, $\prod_{ij}^{p}=1$. This result is applied to the following special cases: (1) F is the general surface of order m; (2) F is the general ruled surface of class m and genus p. In the latter case f is the general curve of order m and genus p, and only the cases $p=0, 1$ are treated explicitly [in the case $p=0$ the same result has been obtained by the reviewer, Amer. J. Math. **58**, 607–619 (1936), in particular, pp. 611–612]. *O. Zariski.*

Zappa, Guido. Invarianti numerici d'una superficie algebrica e deduzione della formula di Picard-Alexander col metodo dello spezzamento in piani. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 121–130 (1946).

This is a continuation of the author's investigations concerning the algebraic and topological invariants of an algebraic surface F, of order n in S_3, which has the special property of being deformable into a set L of n distinct planes, no three coaxial [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 989–1021 (1942); these Rev. **8**, 527; see also the preceding review]. Let γ be the number of double lines of L which are not part of the limit of the double curve of F, let τ be the number of the triple points of L and let x be the number of those triple points at which exactly two of the above γ lines converge. In terms of these numerical characters of L the author calculates the arithmetic genus p_a, the Zeuthen-Segre invariant I and the linear genus ω of F: $p_a=n-\gamma+\tau-1$, $I=3n-2\gamma+6\tau-x-4$, $\omega=9n-10\gamma+6\tau+x+1$. A subdivision of F due to Alexander [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **23**$_2$, 55–62 (1914)] is used in order to express the Euler-Poincaré characteristic of F in terms of the above numerical characters of L, with the known result that the characteristic equals $I+4$. *O. Zariski* (Cambridge, Mass.).

***Conforto, Fabio. Lo stato attuale della teoria dei sistemi di equivalenza e della teoria delle corrispondenze algebriche tra varietà.** Atti Convegno Mat. Roma 1942, pp. 49–83 (1945).

Büke, Macit. Les surfaces quartiques de Segre non-singulières dans l'espace projectif à quatre dimensions. I. Rev. Fac. Sci. Univ. Istanbul (A) **12**, 80–106 (1947). (French. Turkish summary)

This is a study of the incidence properties of sixteen lines, g-lines, in four dimensions, which were discovered by C. Segre [1884] to lie upon the quartic surface Γ common to, and the complete intersection of, two quadric primals (solids) f and ϕ. The work is based on that of P. Du Val and J. A. Todd, particularly the latter's discussion of these g-lines [Proc. London Math. Soc. (2) **32**, 449–487 (1931)], but considerable and detailed advance has here been made by utilizing the continuity principle which has lately been elaborated by B. Segre [Proc. London Math. Soc. (2) **47**, 351–403 (1942); "The Non-Singular Cubic Surfaces," Oxford University Press, 1942; these Rev. **4**, 255, 254].

Incidence properties of the g-lines on the surface Γ, which is rational, are proved to be recognisable in the figure Γ_0, that of four planes in [4], when the quadrics f and ϕ degenerate, by a continuous deformation, into $f_0=x_2x_4$, $\phi_0=x_1x_3$, where $x_0=0, \cdots, x_4=0$ are five primes forming the frame of reference in homogeneous coordinates.

Two conics C_f and C_ϕ are taken, in the planes $x_2=x_4=0$ and $x_1=x_3=0$ respectively, and the tangential form of the quadric f_0 is shown to be that of all the primes which touch the conic C_f. Similarly for C_ϕ. The four lines, from the vertex $(1,0,0,0,0)$, of the simplex of reference lie two by two in these planes, and cut the conics in four pairs of points $P_{\nu\alpha}$ $(\nu=1,2,3,4;\ \alpha=1,2)$. Sixteen of the lines $P_{\nu\alpha}P_{\nu'\alpha'}$ are the images (in this deformation) of the original g-lines. A diagram and a matrix notation make this plain.

Each g-line cuts just five others, and these are skew among themselves. Given one such quintuplet in [4] one such surface Γ is determined (involving 26 parameters). The g-lines are classified into 80 doublets (complementary in 40 pairs), 160 triplets (three g-lines which have a common transversal), 120 quadruplets (80 of one sort, 40 of a second sort), and 16 quintuplets. The triplets form 80 complementary pairs, or double-threes: the quadruplets of the first sort have a common transversal; of the second sort have none. Each quadruplet has a complementary quadruplet (three each for the second sort), and this gives a double-four. This last set of eight leaves a complementary and similarly constituted eight of the sixteen g-lines, so forming a double eight, of which there are ten in all.

The g-lines cut in 40 points, the k-points, which lie by nines on 40 t-planes each containing two g-lines. Two t-planes intersect either in lines (g-lines or else k-lines) or in points, the five c-points, which are vertices of the five cones of the quadric pencil $\lambda f+\mu\phi$. When the surface Γ is projected from a point of itself on to a prime [3] it becomes a cubic surface for which the t-plane projects to a tritangent plane. Many further properties are worked out.

H. W. Turnbull (St. Andrews).

Büke, Macit. Les surfaces quartiques de Segre non-singulières dans l'espace projectif à quatre dimensions. II. Rev. Fac. Sci. Univ. Istanbul (A) **12**, 164–189 (1947).

This work is a direct sequel to the paper reviewed above. It elaborates the group properties of the sixteen g-lines that lie upon each of two general quadric primals (solids) in four dimensions. Among the 16! permutations of such g-lines there are 1920 which alone preserve their incidence properties, as explained in the former paper. These form five conjugate subgroups, each of order 384 for a group G of order $5\times384=1920$. The group G is analysed into all its subgroups, in terms of the geometrical features already given. The investigation is necessarily complicated, but the results are carefully tabulated. A list of eighteen types of subgroups, together with their orders, number of conjugates and geometrical characteristics, is given. There are, for instance, 15 conjugate subgroups of order 16, which leave invariant four mutually permutable double fours of the second kind. Thus the significance of the geometrical classification of the lines into doublets, triplets, etc., becomes apparent. *H. W. Turnbull* (St. Andrews).

Godeaux, Lucien. Sur la construction de surfaces canoniques de l'espace ordinaire. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 288–300 (1946).

Godeaux, Lucien. Construction de surfaces algébriques irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 427–440 (1947).

Godeaux, Lucien. Sur quelques surfaces algébriques irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 457–464 (1947).

Godeaux, Lucien. Recherches sur la construction de surfaces algébriques irrégulières. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 22–32 (1947).

Godeaux, Lucien. Recherches sur la construction de surfaces algébriques irrégulières. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 33–38 (1947).

Godeaux, Lucien. Recherches sur la construction de surfaces algébriques irrégulières. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 67–76 (1947).

In the first two notes the author describes the construction of irregular surfaces ($p_g - p_a > 0$) which represent the involutions of order 3 which belong to the image surface of the unordered pairs of points of a curve containing a cyclic involution of order 3. In the third note he considers the construction in some special cases when the cyclic involution is of order $p > 2$. *D. Pedoe* (London).

Godeaux, Lucien. Sur les surfaces de genres un et de rang trois de l'espace à quatre dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 66–79 (1947).

Godeaux, Lucien. Sur une involution rationnelle appartenant à une réglée elliptique. Math. Notae **6**, 201–212 (1946).

Considérons une cubique plane elliptique L et une homographie h de période trois sur L. La surface F qui représente les couples de points non ordonnés de L appartient à la classe des réglées elliptiques. Tout point de F représente un couple de L dont les homologues réitérés dans h sont représentés sur F par les points constitutifs d'un groupe d'une involution I_3 de F, qui possède trois points unis parfaits et trois points unis non parfaits. La surface F' image de I_3 est rationnelle: l'auteur le démontre par application du théorème de Noether, en mettant en évidence sur F' un faisceau rationnel de courbes rationnelles, puis par application du théorème de Castelnuovo ($p_a = P_2 = 0$). La surface F' est normale dans S_6, d'ordre 9, à sections hyperplanes de genre 4. Elle admet trois points triples et trois points doubles. Les hyperplans passant par les points triples coupent F' suivant trois cubiques gauches. Il existe sur F' deux systèmes linéaires de courbes d'ordre 9: l'un de dimension 5, de degré 6, de genre 2; l'autre de dimension 4, de degré 3, de genre 0. *L. Gauthier* (Nancy).

Godeaux, Lucien. Sur les involutions irrégulières appartenant à une surface irrégulière. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 198–204 (1947).

Nollet, Louis. Sur un théorème de M. De Franchis. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 621–625 (1947).

The author establishes the existence of a pencil of irreducible plane curves of order seven and genus two, whose adjoint system consists of irreducible quartics of genus two. He points out an oversight in the argument which led de Franchis to deny the existence of such a pencil [Rend. Circ. Mat. Palermo **13**, 1–27 (1899)]. *J. A. Todd.*

***Burau, W. Lineare Räume auf quadratischen Hyperflächen.** Ber. Math.-Tagung Tübingen 1946, p. 59 (1947).

Burniat, Pol. Sur des surfaces canoniques triples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 523–539 (1946).

Burniat, Pol. Surfaces canoniques quadruples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 489–507 (1947).

Campedelli, Luigi. Le superficie con i generi uguali all' unità, rappresentabili in infiniti modi sopra un piano doppio. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 105–138 (1940).

Dantoni, Giovanni. Sul genere geometrico delle superficie algebriche con un fascio irrazionale. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 47–56 (1946).

The geometrical genus p_g of an algebraic surface of irregularity q which contains a pencil of genus π is shown to satisfy the relation $p_g \geqq \pi(q-\pi)$, equality holding only when the surface is elliptic or when it is the image of the point-pairs of two curves of respective genera π and $q-\pi$. The proof makes use of the variety of Picard associated with the surface. *J. A. Todd* (Cambridge, England).

Dantoni, Giovanni. Sulle varietà di irregolarità superficiale q contenenti involuzioni $\{C\}$ di irregolarità superficiale π di curve di genere $q-\pi$ e sulle varietà V_n con un gruppo picardiano ∞^p ($p < n$) di trasformazioni birazionali in sè. Ann. Mat. Pura Appl. (4) **24**, 177–194 (1945).

Forbat, N. Sur l'intersection de deux quadriques de révolution. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 374–380 (1947).

Franchetta, Alfredo. Osservazioni sui punti doppi isolati delle superficie algebriche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 283–290 (1946).

Leidheuser, R. W. Sulla ipersuperficie di ordine 4 dello spazio a 4 di dimensioni, generabile con 4 stelle omografiche di iperpiani e la sua rappresentazione su di uno spazio a 3 dimensioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **1**, 320–354 (1940).

Lesieur, L. Sur les espaces linéaires qui sous-tendent une courbe rationnelle normale d'ordre n de l'espace à n dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 611–620 (1947).

Longo, Carmelo. Le rette di una superficie cubica. Boll. Un. Mat. Ital. (3) **2**, 23–24 (1947).

Metelka, J. Sur les points planaires des surfaces cubiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 143–155 (1947).

Nollet, Louis. La réduction à l'ordre minimum des systèmes linéaires de courbes algébriques planes dotés d'un faisceau de bisécantes elliptiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 626–636 (1947).

Nollet, Louis. De la réduction à l'ordre minimum des systèmes linéaires de courbes algébriques planes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 656–677 (1947).

d'Orgeval, Bernard. De quelques variétés de genres 1 possédant des involutions d'ordre premier. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 321–334 (1947).

d'Orgeval, B. De quelques variétés de genres 1 possédant des involutions d'ordre premier. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 381–393 (1947).

d'Orgeval, B. Sur les variétés à trois dimensions de genres 1 contenant une involution d'ordre cinq. I, II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 495–522, 576–599 (1946).

d'Orgeval, B. Sur les variétés à trois dimensions de genres 1, possédant une involution algébrique d'ordre 5. III. Variétés possédant des points unis de troisième espèce. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 9–29 (1947).

d'Orgeval, B. Sur les variétés à trois dimensions de genres 1 possédant une involution cyclique d'ordre cinq. IV. Variétés possédant une courbe de points unis. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 160–172 (1947).

Pompilj, Giuseppe. Su una classe di piani multipli rigati. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 57–74 (1946).

A propos d'un article de Zappa sur le même sujet [Rend. Sem. Mat. Univ. Padova **13**, 41–56 (1942); ces Rev. **8**, 223], l'auteur étudie une classe remarquable et assez générale de courbes, qui sont courbes de branchement pour un plan multiple m-uple. Après avoir étudié une telle courbe comme enveloppe des courbes d'un système algébrique à un paramètre d'indice m, l'auteur examine le cas où ces courbes sont les projections des génératrices rectilignes d'une réglée. La courbe de branchement de cette classe de plans multiples réglés est formée d'une courbe irréductible de classe m de genre p et de ses tangentes singulières de second type avec une multiplicité égale à l'indice ν (c'est-à-dire les tangentes multiples dont la présence entraîne un abaissement de l'ordre de ν unités). Réciproquement toute courbe ainsi décomposée est courbe de branchement d'un plan multiple obtenu en projetant une surface réglée projective de S_3, et si, de plus, $m \geqq 2p+2$ on peut faire en sorte que la réglée ne passe pas par le centre de projection. *L. Gauthier.*

Rollero, Aldo. Sopra una rigata osculatrice ad una superficie algebrica lungo una sua conica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 65–72 (1947).

Vaccaro, Giuseppe. Sopra gli invarianti delle rigate razionali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 234–239 (1946).

By means of the notion of osculating space of a ruled surface along a generator the invariants of a ruled surface F^{2i} of S_{2i+1} are found together with their geometric interpretation. *From the author's summary.*

***Keller, Ott-Heinrich. Eine charakteristische Determinante ebener algebraischer Berührungstransformationen.** Ber. Math.-Tagung Tübingen 1946, pp. 89–91 (1947).

Let $F(x, X) \equiv F(x_1, x_2, x_3; X_1, X_2, X_3) = 0$ be a contact transformation of the two planes e and E. Then X^1 or x^1, respectively, being fixed, $F(x, X^1) = 0$ or $F(x^1, X) = 0$, respectively, will be a curve in e or E, respectively. The equation $\Delta = 0$, which is an equivalent of the Jacobian and which gives the singularities of the point-transformation $F = 0$, is discussed and all cases are enumerated. If F is birational, several formulae connecting the geometrical constants of the transformation are established.

E. Bodewig (The Hague).

Stubban, John Olav. Sur les transformations quadratiques birationelles dans la géométrie de direction. Arch. Math. Naturvid. **48**, no. 3, 37–56 (1945).

This paper is based on the known result that the directed lines (semi-droites) of a plane can be represented on the points of a quadric cone K of space in such a way that the cycles (directed circles) of the plane are represented by plane sections of K. The author defines quadratic transformations of the directed lines of the plane to be such as are derivable from quadric transformations of space which leave K invariant or which transform K into the image K' of the directed lines of another plane. By an exhaustive discussion of the space transformations he finds that the inverses of all the plane quadratic transformations so derived are again quadratic, so that any one of them is derivable from a quadro-quadric transformation of space; and he discusses their fundamental lines. The paper also contains a special study of the involutory cases, and, in particular, of an axial type of transformation in which corresponding lines meet on a fixed line. The last section deals with the transformation of hypercycles. *J. G. Semple* (London).

Stubban, John Olav. Décomposition d'une transformation birationnelle dans la géométrie de direction. Norske Vid. Selsk. Forh., Trondhjem **18** (1945), no. 47, 193–196 (1946).

If T_i ($i = 1, \cdots, \rho$) are a sequence of birational transformations, of orders μ_i, each of which maps the directed lines of a plane on those of a succeeding plane, it is shown that in general the order of the product transformation T is $\kappa = \sum(\mu_i - 1) + 1$. The author finds, conversely, that any T whose order κ has this value can be factorized as above into T_i of order μ_i; and, in particular, T can always be factorized into $\kappa - 1$ quadratic transformations. The method used is to associate every transformation such as T with an ordinary plane Cremona transformation; and this is done by projecting onto a plane each of the Bricard cones whose points image the directed lines of one of the planes involved.

J. G. Semple (London).

***Stubban, John Olav. Sur une classe de transformations birationnelles dans la géométrie de direction.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 329–332. Jul. Gjellerups Forlag, Copenhagen, 1947.

If T is a birational self-transformation of the totality of directed lines of the plane, then T is said to be symmetrical if any general pair of opposite lines of the plane is transformed into another pair of lines which are opposite. The problem of determining such transformations reduces to that of finding the ordinary birational self-transformations of a right circular cone which permute with reflection in the plane through the vertex perpendicular to the axis. The author obtains thus a set of conditions, involving the fundamental lines and other fixed elements, which a general T must satisfy if it is to be symmetrical. *J. G. Semple.*

Stubban, John Olav. Contribution à l'étude des points fondamentaux des transformations crémoniennes de l'espace. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 138–142 (1947).

Derwidué, L. Transformations birationnelles du plan admettant une sextique rationnelle unie. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 311–323 (1946).

Derwidué, L. Sur les transformations birationnelles liées à un faisceau ou une congruence de courbes elliptiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 324–344 (1946).

Derwidué, L. Sur certaines transformations liées à un faisceau de surfaces rationnelles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 152–159 (1947).

Derwidué, L. Sur les involutions et les systèmes de courbes de l'espace. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 307–320 (1947).

Purcell, Edwin J. Some Cremona involutions in n-dimensional space. Univ. Nac. Tucumán. Revista A. **5**, 125–132 (1946).

L'auteur généralise à S_n l'involution plane I_2 de Geiser, déjà étendue à S_3 par Sharpe et Snyder [Trans. Amer. Math. Soc. **20**, 185–202 (1920)]. Au point x_i de S_n on associe le point $\bar{x}_i$ de $\bar{S}_n$ ($i=1, 2, \cdots, n+1$) défini par:

$$a_{ijk}x_i\bar{x}_j=0, \qquad i=1, \cdots, n-1;$$
$$b_{jkl}x_ix_i\bar{x}_j=0, \quad j, k, l=1, \cdots, n+1$$

(en faisant la convention de sommation). L'involution I_2 est celle des couples de points PP' de S_n qui correspondent dans $\bar{S}_n$ à un même point $\bar{P}$. Cette involution est rationnelle, et la transformation birationnelle qu'elle définit est d'ordre $2n^2+n-2$. Le lieu des points fondamentaux de première espèce est une variété à $n-2$ dimensions d'ordre $\frac{1}{2}(n^2+3n+4)$. Les points fondamentaux de seconde espèce forment, d'une part une variété réglée de dimension $n-2$, d'autre part certains lieux de coniques. L'involution I_2 peut être engendrée comme trace sur S_n d'une congruence de coniques de S_{n+1}. *L. Gauthier* (Nancy).

Differential Geometry

Kasner, Edward, and De Cicco, John. Extensions of harmonic transformations. Amer. J. Math. **69**, 575–582 (1947).

Transformations of differential elements of the third order at a point in the plane under the group of arbitrary point transformations, and also under the set of harmonic transformations, are studied. The two theories are compared. *E. F. Beckenbach* (Los Angeles, Calif.).

Kasner, Edward, and De Cicco, John. Conformal perspectivities upon a sphere. Univ. Nac. Tucumán. Revista A. **5**, 203–212 (1946).

The stereographic projection of a sphere S on a plane perpendicular to the diameter of S through the pole of projection is a conformal transformation. This result was established by Ptolemy (about 150 A.D.). Conversely, the only conformal perspectivities on a plane are Ptolemy's projections of a sphere and their limits, the stereographic projections of parallel planes. The converse is now extended to perspectivities on a sphere, and it is shown that if a surface admits a conformal perspectivity on a given sphere, then the surface is also a sphere (or plane). The following results are obtained as corollaries. If a surface admits a conformal perspectivity on a sphere S from the center of S, then the surface is a sphere concentric with S. The only surfaces which admit conformal perspectivities on a sphere S from a point P on S are the spheres tangent to S at P and the planes perpendicular to the diameter of S through P. *E. F. Beckenbach* (Los Angeles, Calif.).

Terracini, Alejandro. Two observations on the preceding paper of Kasner and De Cicco. Univ. Nac. Tucumán. Revista A. **5**, 213–226 (1946). (Spanish)

The author first establishes the results of the preceding paper by methods of synthetic projective geometry. He then, by methods of differential geometry, determines the totality of analytic surfaces S, S', even in the complex domain, between which there exist conformal perspectivities. *E. F. Beckenbach* (Los Angeles, Calif.).

Hartman, Philip. Systems of total differential equations and Liouville's theorem on conformal mappings. Amer. J. Math. **69**, 327–332 (1947).

Liouville's theorem to the effect that every 3-dimensional conformal mapping of class C''' is a Möbius transformation is improved by replacing class C''' by class C''. This is accomplished by proving that a surface of class C'' in Euclidean 3-space is part of a sphere or plane if at each point the normal curvature is independent of direction (known previously for surfaces of class C'''). This depends on the following lemma: if $\alpha(u, v)$ is continuous on a connected domain U in the (u, v)-plane and $x(u, v)$, $y(u, v)$ are of class C' on U with nonvanishing Jacobian, then $\int_C \alpha dx = \int_C \alpha dy = 0$ for every rectifiable Jordan curve C in U if and only if $\alpha =$ constant. *S. B. Myers.*

Ohyama, Takuo. A remark on the extension of Liouville's theorem to a Euclidean space of signature $(+, +, -)$. Tôhoku Math. J. **49**, 139–144 (1943).

The reviewer [Nederl. Akad. Wetensch. Proc. **40**, 700–705 (1937)] proved the following theorem. Every real conformal point transformation in an n-dimensional pseudo-Euclidean space (nondefinite fundamental form) is composed of a motion and a transformation T, where T is either an inversion or a transformation belonging to a class M. In this note the author gives an elementary proof of this theorem using a classical method of Liouville. First he obtains necessary and sufficient conditions for the existence of a transformation which transforms $d\xi^2+d\eta^2-d\zeta^2$ into $H_1^2dx^2+H_2^2dy^2-H_3^2ds^2$, which leads to equations analogous to the Lamé equations. Then these differential equations are solved for $H_1=H_2=H_3$. *J. Haantjes* (Amsterdam).

***Haack, W. Über eine optisch-invariante Schar von Regelflächen eines Strahlenkomplexes.** Ber. Math.-Tagung Tübingen 1946, pp. 70–72 (1947).

From a given vector field in 3-space, $v=v(x)$, a new one is derived by displacing each vector in its own direction an arbitrary distance. The line integral $\int v \cdot dx$ over a closed curve is shown to be invariant under this transformation; several consequences of this fact are derived. *H. Samelson* (Ann Arbor, Mich.).

Woinaroski, Rudolf. Die Kinematik des starren Körpers in einem vierdimensionalen, Euklidischen Raum. Disquisit. Math. Phys. **3**, 119–130 (1943).

Eine starre Bewegung im vierdimensionalen Raum besitzt einen momentanen Drehungspunkt. Diese verschiedenen Punkte bilden im ruhenden sowie im bewegenden Raum eine Kurve C bzw. Γ, welche einander tangieren. Diese Kurven sind aber nicht charakteristisch für die Bewegung. Die Bewegung des der Kurve Γ entsprechenden Vierbeins

gegenüber dem das der Kurve C entspricht, ist eine in der zu beiden Kurven normalen Raum sich abspielende dreidimensionale Bewegung. Bei dieser Bewegung gibt es eine Gerade (δ), die in Ruhe bleibt und den momentanen Drehungspunkt enthält. Die verschiedenen Lagen von (δ) bestimmen in dem sich bewegenden Raum, sowie in dem sich in Ruhe befindenen Raum, je eine Regelfläche Σ und S. In dieser Arbeit wird gezeigt, dass S und Σ für die Bewegung im allgemeinen charakteristisch sind. [Vgl. Woinaroski, Disquisit. Math. Phys. 4, 175–239 (1945); diese Rev. 8, 532.]

J. Haantjes (Amsterdam).

Maeda, Kazuhiko. On some osculating figures of the plane curve. Tôhoku Math. J. 49, 261–301 (1943).

[The author's name was formerly Jusaku Maeda.] A great many problems of the following type are solved. Given a family of plane curves C, to compute loci connected with the subfamily of those C that have at least kth order contact with a given plane curve at a given point. Example: $k=2$. The C's are the images of the curve $y=x^n$ under any similarities (x, y, rectangular coordinates; $n\neq 0$ and fixed). The equations of the following loci are computed: the images of the origin under the similarities corresponding to the various C's of the subfamily, the curve enveloped by the straight lines that are the images of any given straight line through the origin under the same similarities. Other cases: $k=4$, the C's are the Cesàro curves; $k=2$ or 3, the C's are the Cesàro curves of a given index, in particular, they are Ribaucour curves or sine spirals of that index, catenaries, cycloids or conics; $k=2$, the C's are certain conchoids of Nicomedes or of Sluse, Pascal limaçons, involutes of circles or ophiurides. *P. Scherk* (Saskatoon, Sask.).

***Bol, G. Algebraische Flächen in der Differentialgeometrie.** Ber. Math.-Tagung Tübingen 1946, pp. 42–44 (1947).

Attention is called to the fact that there is no sharp separation between algebraic geometry and differential geometry of surfaces. Characterizations of nonruled algebraic surfaces of the third order, and dually of the third class, are given in terms of the defining differential equations. Purely projective geometric characterizations of surfaces of the third order are described, and dually those of the third class. To the latter belong the Roman surfaces of Steiner.

V. G. Grove (Río Piedras, P. R.).

Bompiani, E. Elementi differenziali regolari e non regolari nel piano e loro applicazioni alla curve algebriche piane. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 5, 1–46 (1946).

The purpose of this paper is to carry out some ideas related to certain curvilinear differential elements of a plane curve [cf. E. Bompiani, Rend. Sem. Mat. Fis. Milano 10, 9–36 (1936); Boll. Un. Mat. Ital. (2) 5, 156–168 (1943); these Rev. 7, 392] and to apply these notions to algebraic plane curves and systems of such curves. In the first place the notion of curvilinear elements is extended to curves with a cusp. Thus if the curve is defined by the expansion (*) $x^s=a_{0s}y^s+a_{1s}xy^s+a_{2s}x^2y+a_{0s}y^s+\cdots$ the terms of (*) of the form $a_{ij}x^iy^j$ for which $3i+2j\leqq s$ are said to represent (or define) the cuspidal curvilinear element E_s. Two algebraic curves having the same E_s have at least $s+1$ points in common (at the common cusp). By imposing the conditions that a cubic curve C_3 have the same cuspidal element E_7 as the curve C, a covariant line of C is found. By imposing the conditions that such a two parameter family of cuspidal cubics also have the same E_9 and E_{10}, and choosing the coordinate system in a manner suggested by the analysis, the power series expansion of C may be reduced to a simple canonical form.

Applications of the notions of nonregular curvilinear elements are next made to algebraic plane curves and pencils of plane curves having points of intersection either ordinary or singular points of various kinds on each curve and on each curve of the family. *V. G. Grove.*

Bompiani, E. Sugli invarianti proiettivi di due calotte superficiali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 184–190 (1945).

In recent years there have been extensive investigations, particularly by the author, of the projective invariants and their metric significance of two surface calottes of the second order. [See, for example, the paper of Buzano reviewed below, where there is an extensive bibliography.] In the present paper the author takes up the investigation anew and finds new and simpler projective and metric interpretations of the invariants. In the metric case he finds certain noteworthy properties of invariance with respect to translation and rotation of the calottes relative to one another. Proofs are omitted, having already been presented in lithographed notes of the author to a course in differential geometry [1942–43]. *J. L. Vanderslice.*

Buzano, Pietro. Invarianti proiettivi di una coppia di elementi superficiali del 2° ordine. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 1, 139–162 (1940).

The author gives a definitive discussion of the invariants of two calottes of the second order together with their projective and metric interpretation. He divides the investigation into six cases distinguished by the relative positions of the centers, tangent planes, and principal tangents of the two calottes. Case one, that of general position, has already been studied to some extent. However, the author's methods present it in a simpler and more complete form and give for the first time a geometric interpretation of the second of the two known invariants. Three of the remaining cases are new. In each of them there is just one invariant and the geometric interpretations are notably different. The other two cases, already fully treated by other authors, are mentioned briefly for completeness. *J. L. Vanderslice.*

Gürsan, Feyyaz. Les éléments d'ordre supérieur d'une courbe gauche. Rev. Fac. Sci. Univ. Istanbul (A) 12, 230–236 (1947). (French. Turkish summary)

L'élément d'ordre n d'une courbe plane en un point peut être déterminé par les rayons de courbure de la courbe et des développées successives jusqu'au $(n-2)^e$ ordre [Cesàro, Vorlesungen über natürliche Geometrie, Teubner, Leipzig, 1901, § 22]. Dans un travail antérieur [même Rev. (A) 6, 27–35 (1941); ces Rev. 6, 19] l'auteur étudie le problème analogue pour une courbe gauche en tirant parti de la notion de courbe adjointe. Le présent travail étudie, dans la première partie, le problème de Cesàro pour une courbe sphérique. En se servant des résultats précédamment trouvés il détermine l'élément d'ordre n en fonction des angles de courbure des développées sphériques successives jusqu'à l'ordre $n-2$. (L'angle de courbure est l'arc du grand cercle de la sphère compris entre le point de la courbe et une intersection de son axe polaire avec la sphère.)

Dans la deuxième partie du travail, les résultats du mémoire cité plus haut de l'auteur sont appliqués à quelques courbes gauches particulières. *M. Haimovici* (Jassy).

Arghiriade, E. Sur la courbe caractéristique d'une courbe plane ou gauche. Bull. Sci. École Polytech. Timişoara 11, 62–66 (1943).

The author defines the characteristic curve T_1 of a plane curve as follows. Let M be a point of C and consider the line joining M to the midpoint of the segment connecting two points in which two lines through M making equal angles θ with the tangent to C at M intersect C. The limit of this line as θ approaches zero is called the angular normal D. The locus of the polar of D with respect to the osculating conic of C at M as M varies is the curve T_1. The author also generalizes this definition to skew curves and gives a few results concerning the concept. *J. E. Wilkins, Jr.*

Gheorghiu, Gh. Th. Sur les quartiques à point triple à tangentes distinctes et sur les courbes planes en général. Bull. Sci. École Polytech. Timişoara 12, 52–58 (1945).

L'auteur de cette recension a signalé l'existence dans le plan d'une courbe C^n de l'ordre n, ayant un point multiple de l'ordre $n-1$ à tangentes distinctes, d'une "droite principale" projectivement invariante [G. Fubini et E. Čech, Geometria Proiettiva Differenziale, t. 1, Bologna, 1926, p. 18]. Gheorghiu considère ici le cas $n=4$, et démontre que dans ce cas la droite principale est la polaire du point triple O de la courbe C^4 par rapport à la conique Γ_1 qui est l'enveloppe des droites coupant C^4 en quatre points dans un rapport équianharmonique. La conique Γ_1 appartient au faisceau de coniques défini par la conique Γ_2 qui passe par les six points d'inflexion de C^4 et par la conique Γ_3 qui passe par les huit points de contact de C^4 avec ses quatre bitangentes. Les polaires de O par rapport aux coniques de ce faisceau donnent un faisceau de droites que l'auteur nomme droites principales de C^4. Ce faisceau contient aussi la polaire de O par rapport à la hessienne de C^4. Sur la première droite principale dont on a parlé on peut considérer le rapport anharmonique des points où elle coupe C^4 et celui formé par ses trois points d'intersection avec les trois tangentes à C^4 en O et par le centre du faisceau des droites principales: ce sont les deux invariants projectifs absolus I, R de C^4. On peut appliquer tout cela à une courbe plane quelconque Γ définie par une équation différentielle linéaire du 3e ordre; si O est un point de la courbe, il y a un faisceau de C^4 qui ont dix points communs avec Γ confondus en O; parmi ces C^4 il y en a une dont les trois tangentes en O, prises avec la normale projective de Γ en O, donnent un rapport harmonique; ses droites principales sont les droites principales de Γ en O; la courbure projective de Γ en O s'exprime à l'aide de l'invariant R de la C^4 considérée. *E. G. Togliatti* (Gênes).

Gheorghiu, Gh. Th. Sur une quadrique attachée à une courbe gauche; sur une classe de surfaces. Disquisit. Math. Phys. 2, 99–107 (1942).

In the first part the author determines the quadric with center at O having contact of order 5 with a given curve. The conditions for a curve lying on such a quadric are given. The curves of Tzitzeica form a special example.

In the second part the author determines the surfaces having special asymptotic curves along which the quadrics mentioned in the first part are identical or intersect in two straight lines. *J. A. Schouten* (Epe).

Gheorghiu, Gh. Th. Sur quelques propriétés affines. Disquisit. Math. Phys. 3, 197–206 (1943).

P. Franck [Math. Z. 11, 289–304 (1921)] has given a geometrical interpretation of the affine invariant H of a surface, using the Lie quadric and the total curvature. Here similar interpretations are given for the invariants J [Pick] and S and linear combinations of them by using the quadric of Darboux. Finally some properties of surfaces with a constant mean curvature and surfaces on which J is a constant are dealt with. *J. A. Schouten* (Epe).

Rollero, A. Ancora sugli sviluppi canonici di una superficie nell'intorno di un suo punto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 280–284 (1947).

The coordinates of a generic point on a surface are referred to a tetrahedron whose vertices are that point, arbitrary points on each of the asymptotic tangents, and an arbitrary point not in the tangent plane. An expansion of one nonhomogeneous projective coordinate based on this general tetrahedron in powers of the other two is derived. The coefficients of this expansion are functions of the arbitrary parameters defining the tetrahedron and of the coefficients of the defining differential equations. By imposing conditions on the arbitrary parameters many special power series expansions may be derived, among which the author derives those of Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 25, 149–154 (1937)] and Wilczynski [Trans. Amer. Math. Soc. 9, 79–120 (1908), p. 103]. Attention is called to two new especially simple expansions. *V. G. Grove* (Río Piedras, P. R.).

Anglade, E. Sur les surfaces dont les asymptotiques d'un système appartiennent à des complexes linéaires. Bull. Sci. Math. (2) 70, 173–180 (1946).

Referring the surface in question to its asymptotic curves, then by hypothesis each asymptotic line of one system, say $v=$constant, is contained in a linear complex U_1 which varies with v. The tangents to the curves $v=$constant along $u=$constant generate a ruled surface R_u. The osculating linear complexes to the surfaces R_u along the same curve $v=$constant have a common linear congruence with directrices g, g'. As v varies, g and g' generate two ruled surfaces $(g), (g')$. The osculating linear congruences to these surfaces along g, g' belong to the complex U_1 and the flecnodal tangents of the two surfaces are conjugate with respect to U_1. As a point describes $v=$constant the associated Lie quadrics pass through two fixed lines conjugate with respect to the corresponding complex U_1 and tangent to (g) and (g'). The surfaces (g), (g') are focal nappes of a W-congruence. *J. L. Vanderslice* (College Park, Md.).

Springer, C. E. Union torsion of a curve on a surface. Amer. Math. Monthly 54, 259–262 (1947).

The definition of the geodesic torsion of a curve on a surface involves the notion of the congruence of normals to the surface. The author defines an analogue of geodesic torsion relative to an arbitrary rectilinear congruence. The similar study for the geodesic curvature of a curve on a surface appeared in an earlier paper [Bull. Amer. Math. Soc. 51, 686–691 (1945); these Rev. 7, 172]. A curve C is defined as a "union curve" relative to the given congruence of lines if the osculating plane to C at every point P of C contains the line of the congruence at P. An analytic expression is obtained for the torsion of a union curve in terms of the given rectilinear congruence. Finally, the "union torsion" of any curve C_1 on a surface is defined as the

torsion of the union curve on the surface in the direction of the curve C_1. If the congruence is normal to the surface, the above definition of union torsion coincides with that of geodesic torsion. *A. Fialkow* (Brooklyn, N. Y.).

***Strubecker, Karl. Über die Flächen, deren Asymptotenlinien beider Scharen linearen Komplexen angehören.** Ber. Math.-Tagung Tübingen 1946, pp. 136–139 (1947).

This is almost an excerpt from a paper with the same title [Anz. Akad. Wiss. Wien. Math. Nat. Kl. **78**, 90–94 (1941); these Rev. **8**, 487]. It contains explicitly the representations without integration of the surfaces in question. *J. A. Schouten* (Epe).

Hsiung, Chuan-Chih. Projective theory of surfaces and conjugate nets in four-dimensional space. Amer. J. Math. **69**, 607–621 (1947).

It is known that every nondevelopable surface immersed in a linear space S_4 of four dimensions has on it either a conjugate net or a unique one-parameter family of asymptotic curves. The author studies, by a purely geometric approach, the projective differential geometry of surfaces in S_4 which sustain conjugate nets. A canonical form of the basic system of partial differential equations is geometrically determined, the effects on the system of the transformations which leave invariant the parametric conjugate net N_x are determined and invariants and covariants of the system are obtained. Local power series expansions in nonhomogeneous projective coordinates are determined for a general analytic surface S and for the parametric curves of the conjugate net N_x of S. Hyperquadrics having third order contact with the surface S at a point x are used to define a reciprocal correspondence between certain lines in the tangent plane of S at x and certain planes which pass through x. These lines and planes are called canonical lines and planes, respectively. The canonical lines all pass through a common point called the canonical point and the canonical planes all lie on a quadric hypercone called the canonical quadric hypercone. This hypercone passes through the canonical point and its tangent hyperplane at this point is called the canonical hyperplane. A pencil of hyperquadrics having third order contact with S at x is determined whose members have fourth order contact with one parametric curve of N_x at x and fifth order contact with the other parametric curve at x. Let S_{-1}, S_1 denote the surfaces generated by the Laplace transformed points x_{-1}, x_1 of an ordinary point x of S with respect to the v-, u-curves of the conjugate net N_x. Special classes of conjugate nets are derived by studying the pencil of hyperquadrics having third order contact with S_{-1} (S_1) at x_{-1} (x_1) and contact of first order with S_1 (S_{-1}) at x_1 (x_{-1}). Geometric characterizations are obtained for conjugate nets with equal Laplace-Darboux invariants. *P. O. Bell.*

Terracini, Alejandro. Directrix focal varieties with abnormal absorption in developable varieties. Univ. Nac. Tucumán. Revista A. **5**, 335–361 (1946). (Spanish)

L'auteur apporte une contribution à l'étude des systèmes de droites dans un espace linéaire S_r. Il y envisage des variétés développables V_k à k dimensions (qu'il appelle systèmes Σ), ces variétés étant définies par le fait que le S_k tangent est le même tout le long de l'une quelconque de leurs ∞^{k-1} génératrices. Sur chaque génératrice d'un système Σ il existe $k-1$ foyers qui, lorsque le génératrice engendre Σ, décrivent des variétés focales (auxquelles les différentes génératrices sont généralement tangentes). Ces variétés focales sont en général à $k-1$ dimensions, mais, dans des cas particuliers, on peut avoir des variétés focales W_h à h dimensions ($0 \leqq h \leqq k-2$) en général coupées par les génératrices. De telles variétés focales sont dites variétés focales directrices. Si, sur chaque génératrice, le foyer situé sur une variété focale déterminée compte pour μ, la variété est dite avoir la multiplicité μ, ou encore posséder l'absorption μ [Abs $(w) = \mu$]. Pour une variété focale directrice (w_h) on a Abs $(w_h) \geqq k-h-1$. Lorsque l'on a Abs $(w_h) = k-h$ l'auteur dit que l'absorption de (w_h) est anormale, et l'objet principal de son étude consiste en la recherche de conditions géométriques caractérisant les cas d'absorption anormale. Ces conditions sont fournies par un théorème fondamental, donnant une construction des systèmes Σ à variétés directrices w_h douées d'absorption anormale, à partir de figures spéciales $(T_\alpha{}^h)$ obtenues en menant $\infty^\alpha S_{h+1}$ $(0 \leqq \alpha = r-h-1)$ par chaque ξ_h tangent à une w_h. Comme applications de ce théorème, l'auteur envisage plus spécialement les systèmes Σ de ∞^3 droites admettant une variété focale directrice d'absorption anormale. Il discute de façon approfondie le cas où l'espace S_r dans lequel le système Σ est plongé est d'ordre $r = 5$, et, un peu plus brièvement, celui où $r \geqq 6$. *P. Vincensini* (Besançon).

Levi, B., Santaló, L. A., and De María, C. Enumerative studies on the varieties of contact of the surfaces in a space of n dimensions. Fac. Ci. Mat. Univ. Nac. Litoral. Publ. Inst. Mat. **8**, 3–72 (1946). (Spanish)

Étant donnée une surface dans un espace linéaire à n dimensions (n étant suffisamment grand) et un point simple ordinaire de cette surface, les auteurs envisagent les différentes courbes issues du point tracées sur la surface et étudient, du point de vue énumératif, les variétés engendrées par les différents espaces linéaires osculateurs à ces courbes au point considéré. Pour les variétés engendrées par les espaces $[p]$-osculateurs avec $p \leqq 6$, ils effectuent la détermination complète de la dimension, de l'ordre, et de l'ordre de multiplicité de certaines variétés subordonnées. Les cas $p = 2$, $p = 3$ ont été étudiés par Del Pezzo [Rend. Accad. Sci. Fis. Mat. Napoli **25**, 176–180 (1886)] et par C. Segre [Atti Accad. Sci. Torino. **42**, 1047–1079 (1907)]. La méthode employée dans le travail actuel permet de retrouver les résultats des deux auteurs précédents et de les compléter, notamment en ce qui concerne les cas de dégénérescence des variétés envisagées. Son application aux cas $p \leqq 6$, tout en mettant en lumière la richesse géométrique du sujet et soulignant les difficultés qui surgissent pour les valeurs de p de plus en plus grandes, ouvre la voie aux chercheurs désireux de pousser plus avant dans l'étude entreprise. *P. Vincensini* (Besançon).

Godeaux, Lucien. Sur une suite de quadriques associée à une congruence W. Amer. J. Math. **69**, 490–492 (1947).

L'auteur rappelle d'abord comment, dans des mémoires antérieurs, et notamment dans "La théorie des surfaces et l'espace réglé" [Actual. Sci. Indus., no. 138, Hermann, Paris, 1934], il a associé une suite de quadriques à tout point d'une surface, la première étant la quadrique de Lie, deux quadriques consécutives de la suite se touchant en quatre points qui sont caractéristiques pour les deux quadriques. Récemment, Chenkuo Pa [Amer. J. Math. **69**, 117–120 (1947); ces Rev. **8**, 487] a cherché à définir cette suite de quadriques directement, dans l'espace à trois dimensions, et a indiqué ensuite qu'à tout rayon d'une congruence

W, on peut de même attacher une suite de quadriques. Godeaux montre que la définition directe résulte aisément de remarques faites dans le mémoire cité et que la généralisation à une congruence W avait aussi été donnée dans ce mémoire [pages 21–24]. *M. Decuyper* (Lille).

Vincensini, Paul. Sur une transformation des congruences les unes par les autres. Bull. Sci. Math. (2) **70**, 183–199 (1946).

The author establishes here the results stated in a recent note [C. R. Acad. Sci. Paris **224**, 258–259 (1947); these Rev. **8**, 347] and also some geometric consequences. The line element of the spherical representation of a congruence is taken in the form $ds^2=\lambda(d\mu^2+d\nu^2)$ and then, as a basic step in the proofs, the coefficients of the second fundamental form of a congruence $\Gamma=\mathfrak{R}^{\alpha}_{C_1}(C)$ (obtained from congruence C by rotation through an angle α about the base congruence C_1) are computed and shown to be linear combinations of the coefficients of the second fundamental forms of C and C_1.

Some of the consequences of the basic formulas are as follows. A set $\mathcal{S}$ of congruences with the same mean parameter can be transformed into a set of congruences with the same mean envelope by rotating $\mathcal{S}$ about any one of its own elements through an angle of 90°. If $\Gamma=\mathfrak{R}^{\pi/2}_{C_1}(C)$, where C_1 is a normal congruence, then for each pair of homologous lines of Γ and C the mean parameter of one congruence is equal to twice the distance from the central point of the other congruence to the mean plane of the base congruence. All the congruences with constant nonzero mean parameter can be constructed from one of them by using homothetic transformations having normal congruences as centers. In the space of line congruences E, each equiparametric variety (formed by the different sets of congruences with the same mean parameter on lines with the same spherical representation) is left invariant by the homothetic transformations centered on the manifold itself. *A. Schwartz.*

Roşca, Radu M. Sur certaines classes de congruences W de l'espace elliptique. Disquisit. Math. Phys. **2**, 283–300 (1942).

Dans l'espace elliptique l'auteur étudie les congruences W déduites de la configuration Γ formée par un cycle de quatre courbes liées l'une avec l'autre par une transformation de Bäcklund [J. Math. Pures Appl. (9) **18**, 167–215 (1939); Bull. Math. Soc. Roumaine Sci. **43**, 45–57 (1941); ces Rev. **1**, 84; **7**, 34] Du fait de l'existence dans cet espace des réglées à courbure absolue nulle (réglées de Clifford) il résulte la possibilité d'obtenir trois configurations Γ distinctes, les deux premières étant construites moyennant un théorème de permutabilité. Les binormales d'un couple de courbes en correspondance Bäcklund engendrent les deux nappes focales d'une congruence W. Des propriétés importantes de ce congruence sont développées. *J. Haantjes.*

Delahaye, et Rozet, O. Sur certains couples de surfaces et sur certaines congruences de sphères de Ribaucour. Bull. Soc. Roy. Sci. Liége **15**, 299–305 (1946).

On associe, suivant une certaine loi, à chaque point M d'une surface (M), un point $\bar{M}$ du plan tangent en M. Donc $\bar{M}$ décrit une surface $(\bar{M})$ que l'on suppose non développable. Dans un premier paragraphe les auteurs donnent des conditions pour que les normales à $(\bar{M})$ et (M) en deux points homologues soient orthogonales et pour que la congruence $(M\bar{M})$ soit normale. Dans un paragraphe suivant ils ajoutent aux conditions précédentes celles pour qu'aux asymptotiques de (M) correspondent les lignes de longueur nulle de $(\bar{M})$; la courbure moyenne de $(\bar{M})$ est alors invariante par déformation de (M) conformément à un résultat signalé par Delgleize. Dans un dernier paragraphe ils donnent des cas particuliers explicites de couples de surfaces (M), $(\bar{M})$ associées comme il vient d'être dit, en supposant que (M) est soit un paraboloïde de révolution soit une quadrique à centre de révolution. Dans le premier de ces deux cas il existe un couple de surfaces minima $(\bar{M})$ associées à (M), les deux surfaces du couple constituant les deux nappes de l'enveloppe d'une congruence de sphères de Ribaucour de définition simple. Dans le deuxième cas, il existe un quadruple de surfaces $(\bar{M})$ associées à (M) sur lesquelles les lignes de courbure se correspondent, et qui s'assemblent par couples constituant les enveloppes de deux congruences de sphères de Ribaucour. *P. Vincensini.*

Mayer, O. Sur certaines congruences de droites et suites de Laplace. Ann. Sci. Univ. Jassy. Partie I. **29** (1943), 67–89 (1946).

Ce mémoire généralise l'étude des couples $[K, \bar{K}]$ de congruences stratifiables conjugées et harmoniques, inaugurée par G. Fubini [Ann. Mat. Pura Appl. (4) **1**, 241–257 (1924)], poursuivie par L. Bianchi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **33**$_2$, 369–377 (1924)], et largement développée et approfondie par S. Finikoff [Rend. Circ. Mat. Palermo **53**, 313–364 (1929)]. L'auteur se place dans l'espace projectif S_r $(r\geqq 3)$ et y envisage les congruences K (à surfaces focales distinctes) telles qu'il existe une famille de surfaces (Σ) déterminant des ponctuelles homographiques sur leurs différents rayons: l'ensemble (Σ) est alors dit former une famille R. Les plans tangents aux surfaces d'une famille R aux points où elles coupent un même rayon de K forment une développable Γ qui peut être, soit un faisceau (R est alors dite axiale) soit une développable quadratique (non dégénérée), soit une surface cubique (non dégénérée). La condition pour que K admette une famille R conjuguée conduit l'auteur à distinguer trois cas suivant que Γ est de classe 1, 2 ou 3. Si Γ est de classe 1, K peut être choisie arbitrairement; en outre, si K n'est pas dans un S_3, il suffit, pour qu'une famille R axiale lui soit conjuguée, que le lieu des axes de R soit une congruence au sens restreint (admette deux familles de développables). Si Γ est de classe 2, K est nécessairement telle que les arêtes de rebroussement des développables portées par l'une de ses nappes focales sont coniques. Si Γ est de classe 3, (K) appartient à un type que l'auteur dénomme J (la même dénomination s'appliquant aux deux réseaux focaux), type qu'il caractérise et étudie géométriquement, et dont il signale des cas particuliers donnant lieu à des remarques intéressantes, notamment en ce qui concerne le comportement des suites de Laplace auxquelles elles donnent naissance. Il étend les résultats obtenus aux congruences planes; mentionne un procédé classique permettant de déduire une infinité de congruences J de la connaissance de deux telles congruences, et signale deux problèmes non résolus: celui de la détermination (s'il en existe) des suites J circonscrites à une suite J, et celui de la caractérisation des surfaces d'un S_3 possédant des réseaux J. *P. Vincensini* (Besançon).

Ermolaev, L. The differential geometry of a vector field. The complex of lines specifying a field. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk **3**, 111–124 (1946). (Russian)

To every point of (ordinary) space a vector λ is assigned. The straight lines obtained by moving each vector parallel

to itself in its own direction form (in general) a line complex. Several properties of this system of lines are expressed in terms of λ and its first tensor derivative ϕ. Examples are the case of the "special" complex, where the lines are all tangent to a surface, systems of lines of which the basic vectors belong to the "central" points of the complex, and systems for which the complex degenerates into a family of normal congruences. The author also investigates the case in which the given vector field depends on one parameter, establishing among others the conditions that such a system still determines a line complex. For properties involving higher derivatives of λ the author refers to such papers as W. Haack [Math. Z. 40, 560–581, 703–712; 41, 252–260 (1936)]. *D. J. Struik* (Cambridge, Mass.).

Creangă, Joan. Sulla trasformazione degli intorni del 2° ordine di due punti corrispondenti, nelle corrispondenze puntuali fra due spazi euclidei. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 1, 177–227 (1940).

If two R_n's (flat metric spaces) are mapped on each other the relations between neighborhoods of first order in corresponding points were investigated by Segre [Rend. Circ. Mat. Palermo 60, 224–232 (1936)]. Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 32_2, 376–380 (1923); Ann. Mat. Pura Appl. (4) 1, 259–284 (1924); Boll. Un. Mat. Ital. (1) 3, 49–56, 97–100 (1924); 4, 195–199 (1925)] discussed the relations between neighborhoods of second order for V_2's in R_3 and R_4. This paper deals with the relations between neighborhoods of second order for two R_3's. In the first chapter the elements of second order of two corresponding curves are discussed and the second chapter deals with these elements for corresponding surfaces. The last chapter contains the relations between corresponding principal trihedra in a neighborhood of first order. No use is made of any abridged notation. *J. A. Schouten.*

Rosenfeld, B. Géométrie différentielle des familles de plans à plusieurs dimensions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 11, 283–308 (1947). (Russian. French summary)

The author studies the geometry of m-dimensional planes in Euclidean n-space by means of normalized Grassmann coordinates. These coordinates may be regarded as the components of a vector p in a space R_N^m of $\binom{n+1}{m+1}$ dimensions. Introducing Cartan's affine connection, the geodesics are m-helicoids, i.e., m-planes having the same perpendicular and the same 2-planes containing stationary angles, while the distance from $p(t)$ to $p(0)$ is $p \cdot t$ and the angles are $\omega_\alpha = k_\alpha \cdot t$ $(k_1^2 + \cdots + k_m^2 = 1)$. A linear element is introduced in R_N^m by means of the tangent m-helicoid. The author then studies k-parameter families of m planes: surfaces $(k < n-m)$, congruences $(k = n-m)$ and complexes $(k > n-m)$. This work generalizes that of Segre, Kummer and others for special values of k, m and n.

M. S. Knebelman (Pullman, Wash.).

Hlavatý, V. Zur Lie'schen Kugelgeometrie. II. Kongruenzen (Elementare Eigenschaften). Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 42, 4 pp. (1941).

[Bezüglich Teil I vgl. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. 1941; diese Rev. 7, 483.] Es sei $\mathfrak{p} = \mathfrak{p}(\eta^1, \eta^2)$ eine L-Kugelkongruenz [$\mathfrak{p}$ symbolisiert die homogenen hexasphärischen Koordinaten der L-Kugeln, vgl. Teil I]. Die Determinante des quadratischen Tensors $\mathfrak{a}_{bc} = \mathfrak{p}_b \cdot \mathfrak{p}_c$ $(\mathfrak{p}_b = \partial_b \mathfrak{p})$ sei von Null verschieden. In jedem "Punkte" η_0 bestimmt $\mathfrak{v} = \mathfrak{v}^a \mathfrak{p}_a$ einen Komplex, wozu auch $\mathfrak{p}(\eta_0)$ gehört. Jede L-Kugel von $\mathfrak{v}$, welche $\mathfrak{p}$ berührt, bestimmt mit $\mathfrak{p}$ ein Berührungselement (Punkt und Ebene). Diese auf $\mathfrak{p}$ durch $\mathfrak{v}$ induzierten Mengen werden angegeben für den Fall, dass $\mathfrak{p}(\eta_0)$ eine Kugel, eine Ebene oder ein Punkt ist. Der Projektivwinkel der Komplexe $\mathfrak{v}^a \mathfrak{p}_a$ und $\mathfrak{w}^a \mathfrak{p}_a$ wird definiert und geometrisch interpretiert. Schliesslich wird angegeben wie man die bis auf einen Faktor gegebenen Koordinaten $\mathfrak{p}$ näher festlegen kann.

J. Haantjes (Amsterdam).

Hlavatý, V. Zur Lie'schen Kugelgeometrie. III. Kongruenzen. (Grundgleichungen.) Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 43, 80–86 (1942).

[For part II cf. the preceding review.]

The set of straight lines, the Plücker coordinates of which are given by the equation $\mathfrak{p} = \mathfrak{p}(\eta^1, \eta^2)$ is called a congruence. The complexes $\mathfrak{p}$, $\mathfrak{p}_\alpha = \partial_\alpha \mathfrak{p}$, $\partial_\alpha \partial_\beta \mathfrak{p}$ determine the osculating space σ_2. The dimension n of σ_2 can be 3, 4 and 5. The author introduces some invariant vectors and tensors which appear as coefficients in the Gauss equation and the Weingarten equation belonging to the congruence. Differential relations between these quantities are given for $n = 3$, 4 and 5, from which it is seen that in either case the congruence is determined uniquely by these quantities. *J. Haantjes.*

Hlavatý, V. Zur Lie'schen Kugelgeometrie. IV. Kongruenzen. (Integrable Übertragung.) Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 43, 185–189 (1942).

The complexes $\mathfrak{p}$ and $\mathfrak{v} = e^a \mathfrak{p}_a$ together [see the preceding review] define a linear parabolic congruence. This congruence is singular if (1) $(\mathfrak{v}, \mathfrak{v}) = a_{\alpha\beta} e^\alpha e^\beta = 0$. The vectors e_1^α and e_2^α of the two singular congruences are determined by (1) up to an arbitrarily chosen factor. This factor is further determined by means of the tensors introduced in the paper reviewed above. With these normalized vectors a metric connection $\Gamma_{\beta\gamma}^\alpha$ with vanishing curvature tensor is introduced. This connection is not invariant under the transformation $\mathfrak{p} \to f\mathfrak{p}$. An f-invariant connection is obtained from $\Gamma_{\beta\gamma}^\alpha$ and $\{{\alpha \atop \beta\gamma}\}$. *J. Haantjes* (Amsterdam).

Hlavatý, Václav. On Lie's sphere geometry. II. Congruences. (Elementary properties.) Rozpravy II. Třídy České Akad. 51, 15 pp. (1941). (Czech)

Hlavatý, V. On Lie's sphere geometry. III. Congruences. (Fundamental equations.) Rozpravy II. Třídy České Akad. 52, 28 pp. (1942). (Czech)

Hlavatý, V. On Lie's sphere geometry. IV. Congruences. (Integrable connections.) Rozpravy II. Třídy České Akad. 52, 20 pp. (1942). (Czech)

Summaries of these papers are reviewed above.

Thomas, T. Y. Characteristic coördinates for hyperbolic differential equations in the large. Proc. Nat. Acad. Sci. U. S. A. 33, 242–246 (1947).

In an open simply connected domain D of the (x^1, x^2)-plane there may be defined a system of continuously differentiable functions $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$, where $\det |g_{\alpha\beta}| < 0$ ("hyperbolic" case). The relation $\sum g_{\alpha\beta} \lambda^\alpha \lambda^\beta = 0$ defines two "characteristic directions" at each point of D. There exist two families A and B of "characteristic curves" covering D, which are tangent to the characteristic directions. In a neighborhood of any point of D it is always possible to introduce by a one-to-one continuously differentiable trans-

formation a new coordinate system, in which the curves of A and B become the coordinate curves and hence $g_{11}=g_{22}=0$. The author shows that such a transformation can be found simultaneously for the whole of D, provided the families A and B admit "cross sections" C_A and C_B. Here a regular curve C_A is called a cross section of the family A, if each curve of A intersects C_A in a single point without being tangent. *F. John* (New York, N. Y.).

Weil, André. Sur la théorie des formes différentielles attachées à une variété analytique complexe. Comment. Math. Helv. **20**, 110–116 (1947).

This communication shows how the method of harmonic integrals for obtaining the Picard integrals of the first kind on an algebraic variety, as given in the reviewer's book, The Theory and Application of Harmonic Integrals [Cambridge University Press, 1941; these Rev. **2**, 296], can be extended to give necessary and sufficient conditions for the existence of meromorphic simple integrals with prescribed singularities on a compact analytic variety V of n complex dimensions.

Let $W_1, \cdots, W_k$ be a finite set of open domains covering V and let $\theta_h=\sum_i A_{hi}dz_i$ be a differential form, with coefficients which are meromorphic functions of the local parameters $z_1, \cdots, z_n$, which is defined in W_h and satisfies $d\theta_h=0$. Suppose further that $\theta_h-\theta_k$ is regular in the region common to W_h and W_k. The problem is to find whether there exists a differential form θ on V, which is closed and whose coefficients are meromorphic in the local parameters in any W_h, such that $\theta-\theta_h$ is regular in W (all h). A construction for a form θ is given such that $d\theta$ is regular on V; if it can be shown that $d\theta=0$, θ is meromorphic and satisfies the required properties. The condition that $d\theta=0$ can be expressed as follows. Necessarily, the singular locus of the required integral is the sum of a finite number of analytic varieties $V_1, \cdots, V_l$ of $n-1$ complex dimensions which are algebroid in the neighbourhood of each of their points. Associated with V_i there is a constant c_i such that $\int_C\theta_h=c_i$, whenever the integral is taken round a small circuit C in W_h which encloses a point of V_i. The required condition is that $\sum c_i(V_i\cdot C)=0$ for every 2-cycle C on V, that is, that $\sum c_iV_i$ is homologous to zero.

The results obtained show that a complete theory of Picard integrals of the second and third kinds, as well as of integrals of the first kind, can be obtained by means of harmonic integral theory. The relation of the results to other problems in the transcendental theory of varieties is also discussed.

In connection with the main results of the paper, reference should be made to papers of K. Kodaira [Proc. Imp. Acad. Tokyo **20**, 257–261 (1944); these Rev. **7**, 329], where existence theorems for harmonic integrals with prescribed singularities are given. *W. V. D. Hodge.*

Vranceanu, G. Sur les invariants des espaces de Riemann singuliers. Disquisit. Math. Phys. **2**, 253–281 (1942).

The author considers the geometry of a Riemannian space V_n whose quadratic form is singular. The study is divided into two cases termed holonomic and nonholonomic according as certain Pfaffian systems are completely integrable or not. It is shown that two nonholonomic spaces may be associated with any singular Riemann space. The earlier work of the author on nonholonomic spaces is utilized to develop the geometry of singular Riemann spaces. Special attention is paid to the case in which the coefficients of the quadratic form of V_n form a matrix of rank $n-1$, particularly the subcase $n=3$. *A. Fialkow* (Brooklyn, N. Y.).

***Pinl, M. Über Flächen mit isotropem mittleren Krümmungsvektor.** Ber. Math.-Tagung Tübingen 1946, pp. 121–122 (1947).

The mean curvature vector of a V_2 in R_n is, for $n=3$, zero if it is isotropic. This is the well-known case of minimal surfaces. For $n>3$, surfaces with a nonvanishing mean curvature vector are possible. For $n=4$, to every point of the surface there belongs an isotropic trivector. The case where all these trivectors lie in an R_3 containing the surfaces was studied by Strubecker. The other case and its connexion with Monge's problem is discussed here.

J. A. Schouten (Epe).

Samuel, P. On conformal correspondence of surfaces and manifolds. Amer. J. Math. **69**, 421–446 (1947).

If two V_n's in an R_N are in conformal correspondence such that the tangent R_n's in corresponding points are parallel, there is a linear relation between the corresponding vectors in the R_n's. The method developed by Christoffel for $n=2$, $N=3$ takes the principal axes of this linear operator as coordinate axes. For this paper, the same method is developed for the general case. The local coordinate systems are not always holonomic as in the case $n=2$ and here the anholonomity object $\Omega^{\alpha}_{\beta\gamma}$ comes in. Chapters II and III contain some properties of the linear operator and in chapter IV a review is given of some important manifolds in R_N. Chapter V deals with the case $n=2$; as for $N=3$ the only solutions are the minimal surfaces with isothermic lines of curvature. If the principal directions are all real the problem can be studied completely for all values of n and N as is shown in chapter VI. The V_n's have to belong to a very restricted class. The general case for $n=3$ is discussed in chapter VII; in the holonomic case a complete solution can be given and in the anholonomic case the ds^2 can be given in terms of an arbitrary function of two variables but there are some necessary and sufficient supplementary conditions for the second quadratic form. Chapter VIII gives the complete answer for the holonomic case; the only possible manifolds are the translation manifolds generated by two totally isotropic manifolds and their Cartesian products, and a generalization of the manifolds obtained in chapter VII. Chapter IX deals with the case where the correspondence is not only conformal but also isometric. The correspondence is then a rotation and it is proved that if this rotation is canonically decomposed into rotations in mutually orthogonal planes, the angles of rotation are constant. If $n=3$ or 4 the principal directions are holonomic. Moreover, it is proved that there always exist holonomic principal directions if there are at most two different angles of rotation. *J. A. Schouten.*

Samuel, P. Generalization to a space of Weyl of the problem of the tensors with covariant derivatives zero. Ann. of Math. (2) **48**, 268–273 (1947).

The author considers a Weyl space (i.e., a space with a symmetrical connection, in which a tensor exists for which $\nabla_\gamma h_{\alpha\beta}=a_\gamma h_{\alpha\beta}$) which admits a tensor with covariant derivative zero. If this tensor $g_{\alpha\beta}$ is symmetrical and if $k^{\alpha}_{\beta}=g^{\alpha\gamma}h_{\gamma\beta}$ is locally reducible to the diagonal form, it is proved that a holonomic system exists for which $d\sigma^2=g_{\alpha\beta}du^\alpha du^\beta$ is decomposed into a sum of independent $d\sigma_i^2$ and $h_{\alpha\beta}du^\alpha du^\beta=\sum s_i d\sigma_i^2$. It is shown that in this case a_γ is a gradient. The same result is obtained for a covariant

constant tensor. If a mixed tensor $b^{\alpha}_{\cdot\beta}$ locally reducible to the diagonal form exists for which $\nabla_\gamma b^{\alpha}_{\cdot\beta}=0$ one can distinguish a few cases according to the number of principal values belonging to $b^{\alpha}_{\cdot\beta}$. In the case of one principal value $b^{\alpha}_{\cdot\beta}$ is a constant multiple of δ^{α}_{β}. If there exist at least three different principal values the quantity a_γ is again a gradient and the space is a Riemannian space. The case of only two principal directions, which is more complicated, is treated. [Cf. a former paper, C. R. Acad. Sci. Paris **220**, 160–162 (1945); these Rev. **7**, 176, where the author treats the same problem for Riemannian spaces.] *J. Haantjes.*

Blum, Richard. Sur les tenseurs dérivés de Gauss et Codazzi. C. R. Acad. Sci. Paris **224**, 708–710 (1947).

The author considers the possibility of imbedding an n-dimensional Riemann space V_n in a Euclidean space $R_{n+\nu}$. The following two systems of linear equations in the unknowns $C_{\alpha/\bar{i}\bar{k}\bar{m}}$ and $K_{\alpha\beta/\bar{i}\bar{k}}$, respectively, play a role in the paper:

$$(D')\quad b_{\alpha/\bar{p}\bar{l}}C_{\alpha/\bar{i}\bar{k}\bar{m}}+b_{\alpha/\bar{i}\bar{l}}C_{\alpha/\bar{k}\bar{p}\bar{m}}+b_{\alpha/\bar{k}\bar{l}}C_{\alpha/\bar{p}\bar{i}\bar{m}}-b_{\alpha/\bar{p}\bar{m}}C_{\alpha/\bar{i}\bar{k}\bar{l}}-b_{\alpha/\bar{i}\bar{m}}C_{\alpha/\bar{k}\bar{p}\bar{l}}-b_{\alpha/\bar{k}\bar{m}}C_{\alpha/\bar{p}\bar{i}\bar{l}}=0,$$

$$(E')\quad b_{\beta/\bar{i}\bar{p}}K_{\alpha\beta/\bar{i}\bar{k}}+b_{\beta/\bar{i}\bar{k}}K_{\alpha\beta/\bar{k}\bar{p}}+b_{\beta/\bar{k}\bar{p}}K_{\alpha\beta/\bar{p}\bar{i}}=0,\qquad \alpha,\beta=1,2,\cdots,\nu;\ i,k,l,m,p=1,2,\cdots,n.$$

Here the $b_{\alpha/\bar{i}\bar{k}}$ are the coefficients of the second fundamental form of V_n. Theorems such as the following are proved. (1) If the rank of the matrix of the linear system (E′) is a maximum, then the Ricci equations are a consequence of the Gauss and Codazzi equations of V_n. (2) If $0<\nu\leqq n(n-2)/8$ and if the Gauss equations admit a real solution for the $b_{\alpha/\bar{i}\bar{k}}$ such that the ranks of the matrices of (D′) and (E′) are maxima, then the class of V_n is not more than ν. The second of these theorems with $\nu=1$ was first obtained by T. Y. Thomas [Acta Math. **67**, 169–211 (1936)]. *A. Fialkow* (Brooklyn, N. Y.).

Sen, R. N. On parallelism in Riemannian space. III. Bull. Calcutta Math. Soc. **38**, 161–167 (1946).

[For parts I and II cf. the same Bull. **36**, 102–107 (1944); 153–159 (1945); these Rev. **6**, 188; **7**, 529.] The author considers an n-dimensional Riemannian space containing an orthogonal ennuple of vectors ${}_\tau h^i$. Parallelism with respect to this ennuple is defined by the equations

$$(1)\qquad dV^i+\nabla^i_{ij}V^i dx^j=0,$$

$$\nabla^i_{ij}=\tfrac{1}{2}\sum_\tau {}_\tau h^i\left(\frac{\partial {}^\tau h_i}{\partial x^j}+\frac{\partial {}^\tau h_j}{\partial x^i}\right).$$

A comparison between the classical parallel displacement of Levi-Cività and (1) is made. Autoparallelism relative to (1) is defined and relationships are developed between autoparallel curves and geodesics. Relations are also developed between the curvature tensors defined relative to the metric of the space and relative to the parallelism (1). Geometric interpretations of the new curvature tensors in terms of the geodesic curvature of various autoparallel curves are given. The fundamental invariants of the space are expressed directly in terms of the parallelism (1). *A. Fialkow.*

Petrescu, St. Sur les invariants des hypersurfaces non holonomes V_n^{n-1}. Disquisit. Math. Phys. **3**, 3–84 (1943).

A study on the V_n^{n-1}'s in a V_n, their invariants and their autotransformations. The first chapter is an introduction to the theory of the V_n^{n-1}. The methods of Vranceanu are used. The second chapter deals with the V^{2p}_{2p+1}, its groups and invariants and the transformations that transform the V^{2p}_{2p+1} into itself. The V^{2p+1}_{2p+2} is dealt with in chapter III. Chapter IV contains the special theory of the V_3^2 and V_3^4 and in chapter V the V_4^3 is discussed. *J. A. Schouten* (Epe).

Petrescu, St. Quelques propriétés conformes des sous espaces V_l, dans un V_m de V_n. Bull. Sci. École Polytech. Timişoara **12**, 167–174 (1946).

Soit V_m un espace plongé dans un espace Riemannien V_n. Au point de vue de la géométrie conforme, à chaqu'un d'eux apartient un tenseur conforme de courbure de Weyl $C^{h}_{\mu\nu\varkappa}$, respectivement C^{i}_{jkh}. Pour qu'un espace Riemannien soit conforme à un espace euclidien, il faut et il suffit que le tenseur de Weyl s'annule. Les équations de Gauss dans la géométrie conforme consistent en relations entre ces tenseurs. Dans ces relations se présente encore un tenseur N^{i}_{jkh} qu'on appelle le tenseur conforme de courbure de V_m relatif à V_n [K. Yano, Proc. Imp. Acad. Tokyo **15**, 247–252 (1939); ces Rev. **1**, 175]. Ce dernier tenseur s'annule si V_m est totalement ombiliqué dans V_n, mais la réciproque n'est pas vrai. Outre V_n et V_m l'auteur prend en considération un sous espace V_l de V_m. Alors on a le tenseur conforme de courbure N^{a}_{bcd} de V_l relatif à V_n, le tenseur N'^{a}_{bcd} de V_l relatif à V_m et le tenseur N^{i}_{klm} de V_m relatif à V_n. Pour chaqu'un des espaces on a défini le tenseur de Weyl. L'auteur établit une relation entre les deux tenseurs N^{a}_{bcd}, N'^{a}_{bcd} et les deux tenseurs de Weyl apartenant à V_m et V_n. Etant donné la propriété du tenseur de Weyl et celle des tenseurs de courbure relatif mentionnés plus haut, ces relations permettent de déduire des conditions pour qu'un sous espace totalement ombiliqué dans un autre espace soit conforme à un espace euclidien. L'auteur établit six théorèmes de cette espèce. *O. Varga* (Debrecen).

Haimovici, Mendel. Sur les familles de transformations ponctuelles simplement transitives. Acad. Roum. Bull. Sect. Sci. **28**, no. 3, 5 pp. (1945).

This is a preliminary note. If a set of transformations $X^i=F^i(x^1,\cdots,x^n;a^1,\cdots,a^n)$ in n variables with n parameters is given, in the X_{2n} of X^i, X_h a linear connection can be uniquely fixed by giving some invariant conditions. The quantities of curvature and torsion can be computed. The set is a group if and only if the curvature vanishes. In this case the connection can be brought back to the connection of Cartan and Schouten in the group manifold. A hint is given as to the sets admitting a transitive group in themselves. *J. A. Schouten* (Epe).

Haimovici, M. Les espaces à connexion affine à parallélisme absolu et les systèmes homographiques de courbes. Ann. Sci. Univ. Jassy. Sect. I. **29** (1943), 90–100 (1946).

It is well known that a necessary and sufficient condition for absolute parallelism in an A_n is that $R^{\cdots s}_{\cdots\lambda}$ has the form $P_{\cdots}A^{s}_{\lambda}$. In an A_n of this kind there exists a one-to-one linear correspondence between the directions at every two points. If in an X_{n-1} a system of ∞^{n-1} congruences is given with the property that there exists a linear correspondence between the tangent directions at every two points, there exists an infinity of connections with absolute parallelism for which the curves of these congruences are geodesics. By imposing invariant conditions one of these connections can be singled out. In general, this connection is not symmetrical. A system of ∞^{n-1} congruences as defined can be fixed by giving $n+1$ congruences. There is a link with textile geometry. *J. A. Schouten* (Epe).

Norden, A. La connexion affine sur les surfaces de l'espace projectif. Rec. Math. [Mat. Sbornik] N.S. 20(62), 263–281 (1947). (Russian. French summary)

A subspace X_m of m dimensions of a projective space P_n is said to be normalized if with each point M of X_m are associated two subspaces: P_I of $n-m$ dimensions which has only the point M in common with T_m (the tangent hyperplane to X_m) and P_{II} of $m-1$ dimensions contained in T_m but not containing the point M. These two subspaces are called the normal spaces of the first and second kind. The geometry of X_m is evidently dependent on the selection of P_I and P_{II}. If $x^\alpha=x^\alpha(u^i)$, $\alpha=1, \cdots, n+1$; $i=1, \cdots, m$ are homogeneous coordinates in X_m, P_{II} is determined by the m points of support $y_i^\alpha=\partial_i x^\alpha-l_i x^\alpha$; and if X_σ^α, $\sigma=1, \cdots, n-m$ are the vertices of P_I, then $\partial_j y_i^\alpha=L_{ij}^\sigma y_\sigma^\alpha+p_{ij}x^\alpha+b_{ij}^\sigma X_\sigma^\alpha$. The L_{ij}^σ are coefficients of semi-intrinsic connection and since $L_{(ij)}^\sigma=l_{ij}\delta_0^\sigma$, it follows that $\Gamma_{ij}^\sigma=L_{ij}^\sigma-l_j\delta_i^\sigma$ are the coefficients of a symmetric connection, called the intrinsic connection of X_m. A change in the points of support of P_{II} induces a projective change in Γ. The author considers various normalizations of X_m and also the special cases of a hypersurface, $m=n-1$, and of the space P_n itself, $m=n$. In the latter case the geodesic lines coincide with the straight lines and the intrinsic geometry is then projectively plane. If P_I and P_{II} are reciprocal polars with respect to a hyperquadric Q_{n-1}, the intrinsic connection is then a connection of Weyl in which angles are determined by the projective metric of Q_{n-1}. The subspace X_m of a conformal space M_n is normalized if with each point M of X_m is associated a sphere of $n-m$ dimensions which passes through M and which is completely normal to X_m. The intrinsic geometry is that of Weyl, its angle being defined by the space M_n. If with each point M there is associated another point by means of an inversion, the intrinsic geometry is Riemannian of constant curvature. *M. S. Knebelman.*

Chern, Shiing-Shen. Note on affinely connected manifolds. Bull. Amer. Math. Soc. 53, 820–823 (1947).

If R_{jkl}^i is the affine curvature tensor of an affinely connected n-dimensional manifold M, and $R_{kl}=R_{ikl}^i$, it is stated that the integral of R_{kl} over any 2-dimensional cycle of M is zero. The proof is invalid, for the space of matrices $\|a_{ij}\|$ with $|a_{ij}|>0$ is not simply connected (if $n\geqq 2$), as is claimed. With the notations of page 823, if $n=2$, G/H is not a Euclidean space, but is topologically a Euclidean plane minus a point. *H. Whitney* (Cambridge, Mass.).

Pensov, G. Classification of differential geometric objects of the class v with one component. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 563–566 (1946).

The general method of finding differential geometric objects, first established by Wagner [same C. R. (N.S.) 46, 347–349 (1945); these Rev. 7, 265] is applied in order to find all the differential geometric objects of class v with one component in an X_n. One must find the complete set of invariants for all the subgroups of a simple transitive group. The transformations of the components of the objects are obtained by considering how these invariants transform under transformations of the reciprocal group. The results of Goląb [Math. Z. 44, 104–114 (1938)] are obtained as a special case. It is shown that in X_n ($n\geqq 2$) the class v cannot be greater than 1. For $v=1$ and $n\geqq 3$ the objects are scalars and densities. For $v=1$ and $n=2$ there exists another class of objects transforming as the ratio of the components of a vector. In an X_1 the class v is 1, 2 or 3. For $v=1$ one obtains the vector and the scalar; $v=2$ leads to a geometrical object transforming as the component of an affine connection, whereas a geometrical object transforming as the component of a projective connection has $v=3$. *J. Haantjes* (Amsterdam).

De Donder, Th. Mouvement d'un solide dans un espace riemannien. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 32 (1946), 295–299 (1947).

[For parts I and II cf. the same Bull. (5) 28, 8–16, 60–66 (1942); these Rev. 7, 176.] Let $z=f(x, y; t)$ be the equation of a Euclidean surface imbedded in a Euclidean space of three dimensions at the instant t. The author finds sufficient conditions that $d(\delta s)/dt=0$, where δs is the arc length. *A. Schwartz* (State College, Pa.).

De Donder, Th., et Géhéniau, J. Contribution aux transformations infinitésimales des tenseurs et des spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 32 (1946), 641–655 (1947).

From the authors' summary: Nous montrons que le changement quelconque de variables, $x'^i=F^i(x_0^1, \cdots, x_0^n)$, où les $F^i(x_0^1, \cdots, x_0^n)$ sont n fonctions imposées à priori, peut toujours être engendré au moyen d'une transformation infinitésimale $dx^i/d\tau=X^i(\tau; x^1, \cdots, x^n)$, $i=1, \cdots, n$, où les $X^i(\tau; x)$ peuvent dépendre explicitement du paramètre τ. Nous faisons disparaître ainsi une lacune grave qui existe dans les exposés actuels où l'on suppose que les X^i ne renferment pas explicitement le paramètre τ, conformément à la theorie des groupes continus à un parametre [Sophus Lie]. Nous obtenons aisément la transformation continue des tenseurs ainsi que leurs transformations infinitésimales et leurs transformations finies ou variances. Notre théorie étant indépendante de toute métrique, les formules que nous obtenons pourront s'adapter aux transformations spinorielles. *A. Schwartz* (State College, Pa.).

De Donder, Th. Extension des identités fondamentales de la physique mathématique aux spineurs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 108–118 (1947).

The author follows closely his work in his book [Théorie Invariantive du Calcul des Variations, Gauthier-Villars, Paris, 1935] and arrives at a set of fundamental identities to be satisfied by a density factor M which is a function of the components of one or more tensors. These identities have the same form if M is a function of the components of one or more spinors. *A. Schwartz* (State College, Pa.).

***Heinz, C. Invariantentheorie im nicht-holonomen Basenfeld.** Ber. Math.-Tagung Tübingen 1946, pp. 78–81 (1947).

The author studies the problem of determining the tensor invariants which can be constructed from nonholonomic quantities. Thus, let g_i, g^i, $i=1, \cdots, n$, denote a system of base vectors of the n-dimensional manifold. Further, let α_r^i denote the matrix of the nonholonomic transformation $(\alpha_r^i \neq \partial x^i/\partial \bar{x}^r)$ and let α_k^{*r} denote the inverse matrix. Then the components of the tensor $g_{kl\cdots n}^i=\partial^2 g^i/\partial x^k \partial x^l \partial x^n$ in the nonholonomic coordinates are defined by

$$(1) \qquad g_{\varkappa\lambda\cdots\nu}^\tau=((\alpha_\varkappa^i g_i)_\lambda \alpha_\lambda^j)_\cdots \alpha_\nu^l \alpha_k^{*\tau} g^k.$$

The author claims that if there exist no relations between the $g_{\varkappa\lambda\cdots\nu}^\tau$ (or in the general case between $g_{\varkappa\lambda\cdots\nu\cdots}^\tau$) then no tensor invariants can be constructed. A very brief outline of the proof is indicated. Several other results are stated without proof. *N. Coburn* (Ann Arbor, Mich.).

***Brand, Louis. Vector and Tensor Analysis.** John Wiley & Sons, Inc., New York; Chapman & Hall Ltd., London, 1947. xvi+439 pp. $5.50.

This work not only comprises the standard vector analysis of Gibbs, including dyadics or tensors of valence two and triadics, but gives also a full-length introduction to the algebra of motors, using dual vectors. The scalar and motor products of two motors are defined. It is shown that all the rules of calculation in "real" vector algebra have exact analogues in motor algebra. The entire theory is illustrated by many applications chosen from the rich field amenable to vector and tensor methods. Contents: I. Vector algebra with applications to plane geometry. II. Motor algebra with indications of how to use it in line geometry and mechanics. III. Vector functions of one variable with applications to differential geometry (curves) and kinematics. IV. Linear vector functions. V. Differential invariants. VI. Integral transformations. VII. Hydrodynamics. VIII. Geometry of a surface in 3-space. IX. Tensor analysis, Riemannian geometry. X. Quaternions. A summary is given at the end of each chapter. In a large number of problems (exercises) throughout the book important applications are introduced. As to the notation, bold-face type is used to denote vectors and tensors of higher valence. Only chapter IX is written in index notation. *J. Haantjes* (Amsterdam).

MATHEMATICAL PHYSICS

Quantum Mechanics

Kohn, Walter. Two applications of the variational method to quantum mechanics. Physical Rev. (2) **71**, 635–637 (1947).

(1) The variational method is applied to a perturbed Schrödinger equation, with potentials of Coulomb type only. By using the unperturbed wave function with a variable scale factor as trial function, a lower bound is obtained for the absolute value of the second-order perturbation energy of the ground state, in terms of the unperturbed and first-order perturbation energy. The ground state of a helium-like atom is taken as an illustration.

(2) It is indicated how a well-known minimum property of the sum of the n lowest eigenvalues of a Hamiltonian can be utilized for estimating higher eigenvalues and eigenfunctions. The helium (2^1S) term is treated as an example. *L. Hulthén* (Lund).

Badarau, Gabriel M. Sur le coefficient de transmission d'une barrière de potentiel coulombien. Disquisit. Math. Phys. **1**, 101–116 (1940).

The author solves the Schrödinger equation for a charged particle in a field having a Coulomb potential k/r for r greater than r_0, and having a constant potential for r less than r_0, where r_0 represents the radius of an atomic nucleus. His method is to expand the wave function in a series of Whittaker functions $W_{k,m}$ or $M_{k,m}$ in the region exterior to the nucleus, and in a series of Bessel functions $J_{n+\frac{1}{2}}$ in the interior, and then determine the expansion coefficients by matching at the boundary $r=r_0$. In this way he derives formulas for the coefficient of transmission past such a potential barrier, leading to a theory of the penetration of a nucleus by charged particles of different energy. The formulas obtained are complicated and not well-adapted for numerical computation. *O. Frink* (State College, Pa.).

Badarau, Gabriel M. Contributions à l'étude des barrières de potentiel; niveaux de résonance des particules-α. Disquisit. Math. Phys. **1**, 391–465 (1941).

This is a detailed study of the quantum theory of transmission of charged particles past potential barriers with spherical symmetry, as developed up to 1941, with suggested improvements. It treats also the application of this theory to the question of resonant energy levels for the penetration of atomic nuclei by alpha particles. By the methods of Gamow, of Fowler and Wilson, and others, formulas for the transmission coefficient are derived either by exact or by approximate solution of the Schrödinger equation for different assumed forms of the potential barrier. An improvement by the author is the use of Whittaker functions to obtain an exact solution of the Schrödinger equation for a potential which is of Coulomb type for values of r greater than the nuclear radius [see the preceding review]. Comparison is made between theory and experiment. There is a bibliography with 234 references. *O. Frink* (State College, Pa.).

Viard, Jeannine. Définition d'un repère privilégié en mécanique ondulatoire. C. R. Acad. Sci. Paris **224**, 800–802 (1947).

In noncommutative mechanics, equations are derived which must be satisfied for a frame of reference in motion relative to a fixed frame, if the kinetic energy function is to be "pseudostationary." *O. Frink* (State College, Pa.).

Faure, Robert. Intégrale du premier ordre. C. R. Acad. Sci. Paris **224**, 797–798 (1947).

Corresponding to any first integral of first order of a classical mechanical system with force field derivable from a scalar potential, the author defines a first integral, in the sense of quantum mechanics, of the associated quantum mechanical system, and conversely. *O. Frink.*

Faure, Robert. Intégrale du premier ordre. Correspondance mécanique classique-mécanique ondulatoire. Rôle du champ électromagnétique. C. R. Acad. Sci. Paris **224**, 1209–1210 (1947).

Given a first integral of a classical mechanical system which includes an electromagnetic field, a corresponding first integral of the associated quantum mechanical system is exhibited, on the assumption that a certain condition (VI) is satisfied. If no electromagnetic field is present, no condition such as (VI) is required [see the preceding review]. *O. Frink* (State College, Pa.).

Heisenberg, Werner. Der mathematische Rahmen der Quantentheorie der Wellenfelder. Z. Naturforschung **1**, 608–622 (1946).

In the first part of the paper the author reviews the main facts about the unitary operator S, the so-called characteristic or collision matrix which is to contain all the essential information about the observables describing a quantum mechanical system. He begins with the definition of S in terms of the wave functions of the system and gives various properties of S.

In the second part he discusses two methods of determining S. The first of these consists of using a set of integro-

differential equations for the determination of the wave functions. The S-matrix may then be determined from the solutions of these equations. The second method is the use of a Hermitian operator η related to S through the equations $S=e^{i\eta}$. The author assumes various simple forms for η and discusses them in some detail. He then argues that the correct theory will probably not begin with a simple operator η but will involve η's which are "smooth" functions of the energy and momentum variables and which will contain a universal length of the order of 10^{-13} cm.

In the third part the author compares the proposed S-matrix formulation of quantum mechanics with some aspects of the existing formulation and closes with a set of requirements on the S-matrix. *A. H. Taub.*

Garnir, H. Sur la théorie de la lumière de M. L. de Broglie. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 164–170 (1946).

The author compares the notation and formalism developed by Kemmer [Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939); these Rev. **1**, 95] in the theory of the meson with de Broglie's modifications of the Maxwell equations [Une Nouvelle Théorie de la Lumière, v. 1, Hermann, Paris, 1940] and points out the similarity between the two. He shows how the definition of spin in the latter treatment may be modified to obtain conservation of total angular momentum. *A. H. Taub* (Princeton, N. J.).

Blokhinzev, D. Wave field with the spectrum of masses. Acad. Sci. USSR. J. Phys. **11**, 72–76 (1947).

The problem of describing particles with a discrete or continuous spectrum of states corresponding to different names is discussed. This problem is closely connected with that of linear equations containing derivatives of unlimited scalars. In particular the simplest case, that of a scalar field, is discussed in more detail. *L. Jánossy* (Dublin).

Blohincev, D. I. The wave field with mass spectrum. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 115–120 (1947). (Russian)

A translation is reviewed above.

Klein, O. Meson fields and nuclear interaction. Ark. Mat. Astr. Fys. **34A**, no. 1, 19 pp. (1947).

This paper describes an attempt to incorporate the meson theory describing charged particles into the five-dimensional unified field theory of gravitation and electromagnetism. The usual gauge transformation is identified with the displacement of the zero-point of the fifth coordinate and hence the momentum corresponding to this coordinate is proportional to the electric charge. In order to have the charge of the mesons restricted to integral positive and negative multiples of the elementary charge, the quantum-theoretical wave function of any charged particle must be a periodic function of the fifth coordinate and this periodicity introduces a fundamental length $l_0=hc(2K)^{\frac{1}{2}}/e$, where K is the Einstein gravitational constant. Since the field equations are nonlinear and much too complicated to study, the usual linear approximation is made. It is found that the mass of the mesons is greater than 10^{21} times the electron mass; this requires a certain adjustment of the constants. Also the masses of the multiply charged mesons turn out to be proportional to the absolute value of their charges. Next, Dirac's equation in a gravitational field is written down for the nucleons and the interactions between them caused by the meson field are investigated. One result is that the meson field gives rise to a contribution of $\mu^2/2M$ to the proton mass and none for the neutron. Also a relation is obtained between the magnitudes of the mesic and electric charges which makes the former too small by a factor of 20 compared to experimentally required values. This discrepancy is attributed to the crude approximations made in the calculations. *S. Kusaka.*

Tzou, K. H. Relativistic Hamiltonian system of a particle and relativistic Heisenberg's equation. Chinese J. Phys. **7**, 9–20 (1947).

The author first gives the known Hamiltonian formulation of the relativistic mechanics of a particle in terms of proper time. He then reviews some aspects of the Dirac electron theory after first formulating it in the Heisenberg representation using proper time as a parameter.

A. H. Taub (Princeton, N. J.).

Green, Alex E. S. Self-energy and interaction energy in Podolsky's generalized electrodynamics. Physical Rev. (2) **72**, 628–631 (1947).

The perturbation method of D. J. Montgomery [same Rev. (2) **69**, 117–124 (1946); these Rev. **7**, 404] is used to evaluate the electrostatic self-energy and the electromagnetic interaction energy of charged particles in the electrodynamics developed by B. Podolsky and C. Kikuchi [same Rev. (2) **65**, 228–235 (1944); these Rev. **5**, 277]. It is shown that, to terms quadratic in the charges, the self-energy is finite and negative and that the interaction energy does not contain singularities. *C. Kikuchi* (East Lansing, Mich.).

Eliezer, C. Jayaratnam. The interaction of electrons and an electromagnetic field. Rev. Modern Physics **19**, 147–184 (1947).

The author reviews the status of the classical theory of the point electron as proposed by Dirac and as modified by himself. He also gives a summary of Dirac's quantum electrodynamics obtained by quantizing this theory.

In the first part the author applies the Lorentz-Dirac equations of motion of a charged particle in an external electromagnetic field (which include the radiation damping terms) to the following problems: (1) the motion of a free electron, (2) an electron in a uniform electric field, (3) an electron in a uniform magnetic field, (4) an electron disturbed by a pulse, (5) the hydrogen atom, (6) interaction of two like charges, (7) an electron in the field of a thin charged plate. He suggests, on the basis of the nature of these solutions, that signs of certain terms in the equations of motion do not fit in appropriately. He then discusses two essentially different methods of changing these signs. The first consists in replacing the usual acceleration vector by a vector containing higher derivatives of the velocity vector. He reexamines some of the problems previously discussed and concludes that the new equations of motion lead to more reasonable solutions. The second alternative is to consider that the field of the electron is obtained from a combination of advanced and retarded potentials. A special case of this in which the retarded and advanced potentials occur symmetrically is called a third alternative. This special case has also been considered by J. A. Wheeler and R. P. Feynman [same Rev. **17**, 157–181 (1945)].

The remainder of the paper is devoted to the Dirac quantum electrodynamics. After a brief review of the quantization of the classical theory the author considers the following problems: (1) interaction of an electron and a

radiation field, (2) hydrogen-like atoms, (3) the interaction of two moving particles, (4) the Compton effect and radiation damping. In (2) he shows that all divergent terms in the wave function of an electron in the Coulomb field of a nucleus and its own radiation field are eliminated. One would then expect that a finite displacement of the spectrum lines of hydrogen-like atoms would be obtained on the basis of Dirac's electrodynamics. This calculation is not reported but should be carried out in view of the experiments of Lamb and Retherford [Physical Rev. (2) 72, 241–243 (1947)]. *A. H. Taub* (Princeton, N. J.).

De Donder, Th., et Géhéniau, J. Le modèle-champ de la mécanique ondulatoire de Dirac. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 31 (1945), 301–303 (1946).

Bodiou, Georges. Signification possible de la limitation à 4 du nombre des composantes de la fonction d'onde de Dirac. C. R. Acad. Sci. Paris 224, 798–800 (1947).

The operators which represent the three components of velocity in the Dirac theory obey the same commutation rules as the Dirac operators α_1, α_2, α_3. The author concludes from this fact alone that the components of the Dirac wave function must be limited in number to four. *O. Frink.*

Møller, C. General properties of the characteristic matrix in the theory of elementary particles. II. Danske Vid. Selsk. Mat.-Fys. Medd. 22, no. 19, 46 pp. (1946).

The author continues his investigation of the S-matrix [same Medd. 23, no. 1 (1945)]. General properties are those which are independent of the particular Hamiltonian. It is assumed that they will also be present if the S-matrix cannot be derived from a Hamiltonian. In the present paper the author studies systems composed of two particles of equal mass. Since the formalism is Lorentz invariant it is sufficient to consider that frame of reference in which the center of gravity is at rest. Originally, the S-matrix was only defined for unbound states (i.e., in the continuous spectrum where the relative kinetic energy is positive for infinitely large separation of the two particles). The energies of bound states, however, as well as the decay constants of radioactive systems, are also observable quantities and must be derivable from a complete theory. The investigation is based on the following remark, attributed to H. A. Kramers. In the continuous spectrum, the wave function has, for large values of the relative distance r, the asymptotic form

$$\text{(A)} \qquad \psi \sim Cr^{-1}\{(-1)^{l+1}e^{-ikr}+e^{ikr}S^0\}\,Y_{lm},$$

where S^0 is an eigenvalue of the matrix S (S^0 and C depend on l and on k or W^0, W^0 being the total energy of the two particles in the center of gravity system). For the known quantum mechanical systems, S^0 is an analytic function of W^0 so that, by analytic continuation, one obtains similar equations for arbitrary (in general complex) values of W^0. If W^0 is real, but smaller than the sum of the rest masses of the two particles, k is imaginary, $k=-i|k|$, and (A) describes a bound state only if S^0 vanishes.

The assumption that S depends analytically on W is then explicitly introduced, and the author carefully investigates the analytical continuation of the matrices in question. Integrals involving the functions δ, δ_+ and δ_- are defined by suitably chosen contour integrals in the complex plane. The equations $SS'=S'S=1$ remain valid; they are, however, interpreted as follows: S' is the analytic continuation of the S-matrix's adjoint which has been defined for positive values of the kinetic energy. In particular, the equation (B) $S^0S^{0\prime}=1$ holds for the eigenvalues but S^0 no longer has the absolute value 1. The following general condition is derived. At a value of W^0 which corresponds to a bound state we have, for a fixed value of l, the inequality $(-1)^l dS^0/dW^0>0$. This shows that bound states appear only at simple zeros of S^0. For radioactive decay, W assumes complex values $W^0=E_1-\frac{1}{2}i\lambda$. It is shown that S^0 cannot vanish if both E and λ are positive. Instead of (A) one may write for the time-dependent wave function

$$\text{(A*)} \qquad \psi \sim C'e^{-iW^0t/\hbar}r^{-1}\{(-1)^{l+1}e^{-ikr}S^{0\prime}+e^{ikr}\}\,Y_{lm}$$

and it is seen that a decay process corresponds to a zero of $S^{0\prime}$, i.e., to a pole of S^0 (cf. (A)). The general results are illustrated by a few examples. The Coulomb potential requires a special procedure (cutting off at large distances); as is well known, it decreases so slowly that the asymptotic relation (A) no longer holds.

Another result of general interest is this: a given S-matrix is either consistent with infinitely many different Hamiltonians or it cannot be derived from a Hamiltonian at all. *V. Bargmann* (Princeton, N. J.).

ter Haar, D. On the redundant zeros in the theory of the Heisenberg matrix. Physica 12, 501–508 (1946).

The S-matrix introduced by Heisenberg [Z. Physik 120, 513–538, 673–702 (1943); these Rev. 4, 292] is discussed for the two-body problem. As it is an analytic function of the total energy W of the system it can be continued analytically for complex W from which it can be shown [cf. the preceding review] that energies W_0 of bound states of the system must satisfy the condition (1) $S(W_0)=0$. Conversely, however, (1) yields, besides the eigenvalues of the problem, redundant values of W_0 for, e.g., a potential (2) $-Be^{-r}$ [Ma, Physical Rev. (2) 69, 668 (1946)]. It is shown that these redundant zeros of S for the potential (2) do not satisfy Møller's condition $(dS/dW)_{W=W_0}>0$ and can therefore be discarded. If the potential is cut off at an arbitrarily large but finite distance, the redundant zeros do not occur. It is conjectured that the redundant zeros occur when the continuum wave function behaves at infinity like $r^{-1}\exp(ikr+c\log r)$. *A. Pais* (Princeton, N. J.).

Jost, Res. Bemerkungen zu der vorstehenden Arbeit. Physica 12, 509–510 (1946).

The zero points of the Heisenberg S-matrix are discussed for a two-body problem with potential $\sum_{n=1}^{n}a_ne^{-nr}$ which is assumed to converge for $\Re(r)>-\delta$, $\delta>0$. It is shown that redundant zeros occur when the continuous wave function behaves at infinity like $r^{-1}\exp(ikr+c\log r)$ in agreement with ter Haar's conjecture [see the preceding review]. However, the condition $(dS/dW)_{W=W_0}>0$ does not now in general suffice to discard the redundant zeros. The nature of these redundant zeros is therefore still unclear. Potentials of the type $\sum_{n=0}^{n-2}a_nr^n/\sum_{n=0}^{n}b_nr^n$, $b_n=1$, do not yield redundant zeros. *A. Pais* (Princeton, N. J.).

Géhéniau, Jules, et van Isacker, Jacques. Divers modes de changements d'états du corpuscule de spin un et zéro. C. R. Acad. Sci. Paris 225, 98–100 (1947).

The author writes the wave function ψ of a particle of spin zero or one as a two-index four-component spinor. An interaction operator representing changes of states of spin is written as a four-index four-component spinor. The wave functions are expanded in terms of a basis formed from the Dirac matrices γ_α ($\alpha=1, 2, 3, 4$) and their products;

the interaction operator is assumed to be of the form

$$H=N_a^+\Gamma_+^a+N_{ab}^+\Gamma_+^{ab}+N_{abc}^+\Gamma_+^{abc}+N_a^-\Gamma_-^a+N_{ab}^-\Gamma_-^{ab}+N_{abc}^-\Gamma_-^{abc}+N_5^-\Gamma_-^5,$$

where $\Gamma_\alpha^\beta=\gamma_\beta\times 1\pm 1\times\gamma_\beta$, the cross denotes the direct product, and γ_β ranges over the fifteen linearly independent matrices formed from the γ_α and their products. The expression $\psi^*\Gamma\psi$ is then formed and the various nonvanishing terms are noted and associated with changes in states of spin of the particle described by the wave function ψ.

A. H. Taub (Princeton, N. J.).

Van Isacker, J. Sur la décomposition des équations d'une particule de spin 3/2. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 554–563 (1947).

Lubański [Physica **9**, 310–324 (1942), in particular, p. 316; these Rev. **5**, 223] has pointed out that the spinors appearing in the reduced equations of a particle with spin give irreducible representations in a five-dimensional space. Using this property, simple equations for a particle of pure spin 3/2 are found, and the equations for a particle of spin 3/2 are decomposed to give various particles of maximum and minimum mass and spin 1/2 and 3/2. *A. Schwarts.*

Arnous, Edmond. Un aspect des difficultés du formalisme des particules de spin supérieur à 1/2. C. R. Acad. Sci. Paris **224**, 795–797 (1947).

A reason is given why the usual methods of deriving probability densities having the correct relativistic variance for particles of spin greater than ½ are likely to lead to expressions which are not positive definite. *O. Frink.*

Slansky, Serge. Sur l'invariance relativiste de l'opérateur d'interaction de deux corpuscules. C. R. Acad. Sci. Paris **224**, 631–633 (1947).

The author derives conditions which must be satisfied for the interaction term in the wave equation of two particles, if the equation is to be relativistically invariant.

O. Frink (State College, Pa.).

Garnir, H., et Toussaint, J. Sur la théorie des valences dirigées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **32** (1946), 508–516 (1947).

Nous nous proposons de montrer que la théorie des groupes permet de retrouver les matrices introduites par L. Pauling, dans sa théorie des valences dirigées.

Extract from the paper.

Thermodynamics, Statistical Mechanics

Popoff, Kyrille. Observations sur la notion de l'entropie et sur quelques formules qui en découlent, concernant l'énergie interne. Ann. Physique (2) **2**, 195–208 (1947).

In this paper the Joule-Thompson formula is derived as a condition of integrability, and the functional relation is determined among all possible definitions of entropy (corresponding to the various possible evaluations of the integrating factor μ). It is shown that

$$u(t_2, v_2)-u(t_1, v_1)=\frac{t_1t_2}{t_1-t_2}\oint\frac{p(t, v)}{t}dv,$$

this formula being particularly suitable for liquids, while Kirchhoff's analogous formula is particularly suitable for gases. A similar formula is obtained with t and p as independent variables. *C. C. Torrance* (Annapolis, Md.).

Popoff, Kyrille. Sur la notion de l'entropie et sur quelques formules qui en découlent. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 271–288 (1946). (Bulgarian. French summary)

Dans ce mémoire nous avons taché de délimiter la partie purement mathématique des hypothèses physiques dans la définition de la notion de l'entropie. Le chemin que nous avons suivi nous a permis de déduire quelques formules concernant les différences $u(t_2, v_2)-u(t_1, v_1)$ et $u(t_2, p_2)-u(t_1, p_1)$ de l'énergie interne d'une masse homogène, formules correspondant aux formules de Kirchhoff et que nous pensons nouvelles. Nos considérations portent sur une intégrale double, équivalente à l'intégrale linéaire le long d'un contour fermé. *Author's summary.*

Popoff, Kyrille. Sur un théorème fondamental de Nernst de la thermodynamique. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 369–379 (1946). (Bulgarian. French summary)

The author shows that, for the third (Nernst) fundamental law to be valid, it is necessary that $\lim_{t\to 0} v_t(t, p)=\lim_{t\to 0} p_t(t, v)=\lim_{t\to 0} c_p(t)=\lim_{t\to 0} c_v(t)=0$. An equation of state meeting these conditions is $pv=Rt^{1+\alpha}$, where α is positive and arbitrarily small. *C. C. Torrance.*

De Donder, Th. Sur le temps thermodynamique de Prigogine. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 560–565 (1946).

Let $dq'(t)/dt=T^{-1}dQ'(t)/dt$ denote the entropy created within a mass-closed system per unit of time t. If the time scale t is transformed by a relation $\tau=\phi(t)$ such that $(d\tau/dt)_0=1$, then

$$(*)\qquad d\tau/dt=\frac{(dq'(t)/dt)/(dq'(t)/dt)_{t=t_0}}{(dq'(\tau)/d\tau)/(dq'(\tau)/d\tau)_{\tau=\tau_0}}.$$

Let the function $\tau=\phi(t)$ be determined by the condition that the denominator of (*) is identically 1, and assume that $q'(t_0)=0$. Then $q'(t)=\alpha(\tau-\tau_0)$, showing the physical meaning of the thermodynamic time τ. If the given system undergoes a reaction with speed $v=a_0e^{-\lambda t}$, so that $dq'(t)/dt=Av/T$, and if A/T is constant relative to t, then (*) has the approximate solution $\tau=(t_0+\lambda^{-1})\log(t+\lambda^{-1})/(t_0+\lambda^{-1})+t_0$ for $\lambda t\ll 1$. This relation is similar to Milne's formula $\tau=t_0\log(t/t_0)+t_0$ for "dynamic" time τ. *C. C. Torrance.*

Adams, Elliot Q. A table of logarithmic corrections to the Wien radiation law. J. Opt. Soc. Amer. **37**, 695–697 (1947).

The author expands Planck's radiation formula in an exponential series and shows how successive terms of the series may be calculated from a table of $c_2\log e/(\lambda T)$ as a function of $-\log(1-10^{-c_2\log e/\lambda T})$. *C. C. Torrance.*

Dufour, L. Sur les transformations adiabatiques et isobariques de l'air atmosphérique humide en système fermé. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 540–546 (1946).

Prigogine, I., et Defay, R. La stabilité des transformations azéotropiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 48–63 (1947).

The production of entropy accompanying a transformation is an extreme when, and only when, the transformation

is azeotropic. This result is used to show that azeotropic vaporization (condensation) is stable when the azeotropy is negative (positive) and unstable when the azeotropy is positive (negative). *C. C. Torrance* (Annapolis, Md.).

Montroll, Elliott W. Averages over normal modes of coupled oscillators with application to theory of specific heats. Quart. Appl. Math. 5, 223–227 (1947).

The computational procedure described in this paper employs the following facts. (1) Let ν denote the frequencies of normal modes of a system of coupled elements. Then $\sum\nu^{2m}$ is a known function of the trace of the mth power of the characteristic matrix B. (2) Let ν_L denote the largest value of ν, let $x=\nu/\nu_L$ and let $\alpha(x)$ denote the number of values of ν/ν_L which are less than x. Then the moment $\mu_{2m}=\int_{-1}^{1}x^{2m}d\alpha(x)$ is a known function of trace B^m. (3) Let $f(\nu)$ be a given function to be summed over values of ν, such as occurs in the theory of specific heat. Let $F(\nu)=f(\nu)$ for $\nu\geqq 0$, and $F(\nu)=f(-\nu)$ for $\nu<0$. Then the given summation can be written as $\frac{1}{2}\int_{-1}^{1}F(x\nu_L)d\alpha(x)$ and this integral can be approximated effectively by the Gauss-Jacobi quadrature formula from the knowledge of the moments μ_{2m}.

The author gives a numerical illustration and states that analogous methods can be used in evaluating averages over characteristic values of linear operators.

C. C. Torrance (Annapolis, Md.).

Wisniewski, Félix Joachim. La statistique quantique. C. R. Acad. Sci. Paris 223, 313–315 (1946).

L'auteur obtient les répartitions de Maxwell, Planck, Fermi, commes des solutions stationnaires d'une certaine équation différentielle dans l'espace de phase à six dimensions de Gibbs-Boltzmann. La différence entre la théorie quantique et la théorie classique consiste en l'ordre d'approximation relativement à la vitesse hydrodynamique de propagation. *C. J. Bouwkamp* (Eindhoven).

Bogolubov, N. Expansions into a series of powers of a small parameter in the theory of statistical equilibrium. Acad. Sci. USSR. J. Phys. 10, 257–264 (1946).

The author considers a system of N identical monatomic molecules in a volume $V=Nv$, the configuration of any molecule being uniquely determined by its three Cartesian coordinates. The forces between the molecules are assumed to be central. Let F_s represent the distribution function (of order s) determining the probability density for a given group of s molecules to occupy specified positions in configuration space. Integrodifferential equations are derived for F_s in classical statistical equilibrium. These are solved approximately in a formal manner, by expansions in ascending powers of $1/v$. Thus, avoiding the usual combinatorial analysis [cf. J. E. Mayer and E. Montroll, J. Chem. Phys. 9, 2–16 (1941)], the author easily arrives at the Ursell-Mayer expansion of the equation of state. The general theory is applied to the case of the Coulomb interaction force. Special attention is paid to the binary function F_2 in its relation to the problem of the condensed state.

C. J. Bouwkamp (Eindhoven).

Bogolubov, N. Kinetic equations. Acad. Sci. USSR. J. Phys. 10, 265–274 (1946).

This paper is an extension of the methods used before in the steady-state problem [cf. the preceding review], in that the time variation of the distribution functions F_s, now defined in phase space, is investigated.

C. J. Bouwkamp (Eindhoven).

Bogolubov, N. Expansions into a series of powers of a small parameter in the theory of statistical equilibrium. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 681–690 (1946). (Russian. English summary)

Bogolubov, N. Kinetic equations. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 691–702 (1946). (Russian. English summary)

Translations are reviewed above.

Bogolyubov, N. N. A contribution to the theory of superfluidity. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] 11, 77–90 (1947). (Russian. English summary)

The present paper is a study of the phenomenon of superfluidity on the basis of the degeneration theory of a nonideal Bose-Einstein gas. Employing the method of second quantization, together with an approximate procedure, the author proves that in the case of small interaction between the molecules the weakly excited states of the gas can be depicted as an ideal Bose-Einstein gas, consisting of some "quasiparticles" representing elementary excitations. These quasiparticles are a collective effect and cannot be identified with the individual molecules. It is shown that the special form of the dependence of the energy of the quasiparticles on their momentum is directly associated with the property of superfluidity. *Author's summary.*

BIBLIOGRAPHICAL NOTES

Nachrichten der Mathematischen Gesellschaft in Wien.

Volume 1, number 1 is dated June, 1947. It contains a list of members of the Mathematische Gesellschaft in Wien, personal notes, abstracts of lectures given before the Gesellschaft and abstracts of Austrian mathematical publications. Three numbers a year are to be published and will contain abstracts of foreign as well as Austrian publications. The Nachrichten will be sent free on request to interested mathematicians; the address is Wien IV. Karlsplatz 13 (Technische Hochschule).

Disquisitiones Mathematicae et Physicae.

Volume 1, number 1 appeared in 1940. Five volumes have been published. The contents are in English, French, German or Italian. This is a publication of the Institutul Regal de Cercetări Ştiinţifice al României, Bucureşti (formerly Institutul de Cercetări Ştiinţifice Regele Carol II).